Construction Economics & Finance

Module 1

Engineering economics

Lecture-1

Basic principles:–

Time value of money:

The time value of money is important when one is interested either in investing or borrowing the money. If a person invests his money today in bank savings, by next year he will definitely accumulate more money than his investment. This accumulation of money over a specified time period is called as time value of money.

Similarly if a person borrows some money today, by tomorrow he has to pay more money than the original loan. This is also explained by time value of money.

The time value of money is generally expressed by interest amount. The original investment or the borrowed amount (i.e. loan) is known as the **principal**.

The amount of interest indicates the increase between principal amount invested or borrowed and the final amount received or owed.

In case of an investment made in the past, the total amount of interest accumulated till now is given by;

Amount of interest = Total amount to be received – original investment (i.e. principal amount)

Similarly in case of a loan taken in past, the total amount of interest is given by;

Amount of interest = Present amount owed – original loan (i.e. principal amount)

In both the cases there is a net increase over the amount of money that was originally invested or borrowed.

When the interest amount is expressed as the percentage of the original amount per unit time, the resulting parameter is known as the ***rate of interest*** and is generally designated as ‘*i*’.

The time period over which the interest rate is expressed is known as the ***interest period*.** The interest rate is generally expressed per unit year. However in some cases the interest rate may also be expressed per unit month.

Example: 1

A person deposited Rs.1,00,000 in a bank for one year and got Rs.1,10,000 at the end of one year. Find out the total amount of interest and the rate of interest per year on the deposited money.

Solution:

The total amount of interest gained over one year = Rs.1,10,000 - Rs.1,00,000 = Rs.10,000

The rate of interest *'i'* per year is given by;

$$i(\%) = \frac{Rs.10{,}000}{Rs.1{,}00{,}000} \times 100 = 10\%$$

Similarly if a person borrowed Rs.1,50,000 for one year and returned back Rs.1,62,000 at the end of one year.

Then the amount of interest paid and the rate of interest are calculated as follows;

The total amount of interest paid = Rs.1,62,000 - Rs.1,50,000 = Rs.12,000

The rate of interest *'i'* per year is given by;

$$i(\%) = \frac{Rs.12{,}000}{Rs.1{,}50{,}000} \times 100 = 8\%$$

Simple interest:

The interest is said to simple, when the interest is charged only on the principal amount for the interest period. No interest is charged on the interest amount accrued during the preceding interest periods. In case of simple interest, the total amount of interest accumulated for a given interest period is simply a product of the principal amount, the rate of interest and the number of interest periods. It is given by the following expression.

$$I_T = P \times n \times i$$

Where

I_T = total amount of interest

P = Principal amount

n = number of interest periods

i = rate of interest

Simple interest reflects the effect of time value of money only on the principal amount.

Compound interest:

The interest is said to be compound, when the interest for any interest period is charged on principal amount plus the interest amount accrued in all the previous interest periods. Compound interest takes into account the effect of time value of money on both principal as well as on the accrued interest also.

The following example will explain the difference between the simple and the compound interest.

Example: 2

A person has taken a loan of amount of Rs.10,000 from a bank for a period of 5 years. Estimate the amount of money, the person will repay to the bank at the end of 5 years for the following cases;

a) Considering simple interest rate of 8% per year

b) Considering compound interest rate of 8% per year.

Solution:

a) Considering the simple interest @ 8% per year;

$$\text{The interest for each year} = \text{Rs.}10{,}000 \times 0.08 = \text{Rs.}800$$

The interest for each is year is calculated only on the principal amount i.e. Rs.10,000. Thus the interest accumulated at the end of each year is constant i.e. Rs.800.

The year-by-year details about the interest accrued and amount owed at the end of each year are shown in Table1.1.

Table 1.1 Payment using simple interest

End of year (EOY)	Amount of interest (Rs.)	Total amount owed (Rs.)
1	800	10,800
2	800	11,600
3	800	12,400
4	800	13,200
5	800	**14,000***

***** *The amount repaid to the bank at the end of year 5 (since the person has to repay at EOY 5).*

The total amount owed at the end of each year using simple interest is graphically shown in Fig. 1.1.

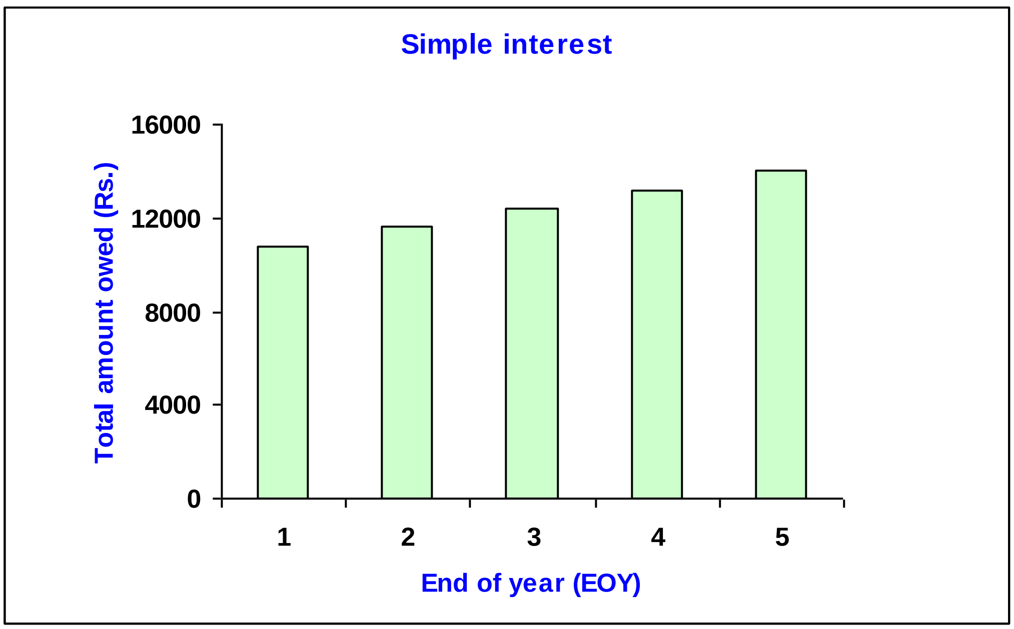


Fig. 1.1 Total amount owed (using simple interest)

b) Considering the compound interest @ 8% per year;

The amount of interest and the total amount owed at the end of each year, considering compound interest are presented in Table 1.2.

Table 1.2 Payment using compound interest

End of year (EOY)	Amount of interest (Rs.)	Total amount owed (Rs.)
1	800.00	10,800.00
2	864.00	11,664.00
3	933.12	12,597.12
4	1007.77	13,604.89
5	1088.39	**14,693.28****

** ***The amount repaid to the bank at the end of year 5.***

The year-by-year values of amount of interest and the total amount owed (as shown in Table 1.2) are calculated as follows;

$$\text{Amount of interest accumulated at the end of Year } 1 = \text{Rs.}10{,}000 \times 0.08 = \text{Rs.}800$$

$$\text{Total amount owed at the end of Year } 1 = \text{Rs.}10{,}000 + \text{Rs.}800 = \text{Rs.}10{,}800$$

$$\text{Amount of interest accumulated at the end of year } 2 = \text{Rs.}10{,}800 \times 0.08 = \text{Rs.}864$$

$$\text{Total amount owed at the end of Year } 2 = \text{Rs.}10{,}800 + \text{Rs.}864 = \text{Rs.}11{,}664$$

Similarly the interest amount and the total amount owed at the end of year 3, year 4 and year 5 can be calculated in the same manner.

From these calculations it is clear that, in case of compound interest the interest for each year is calculated on the principal amount plus the interest amount accumulated till that period.

The total amount owed at the end of each year using compound interest is also graphically shown in Fig. 1.2.

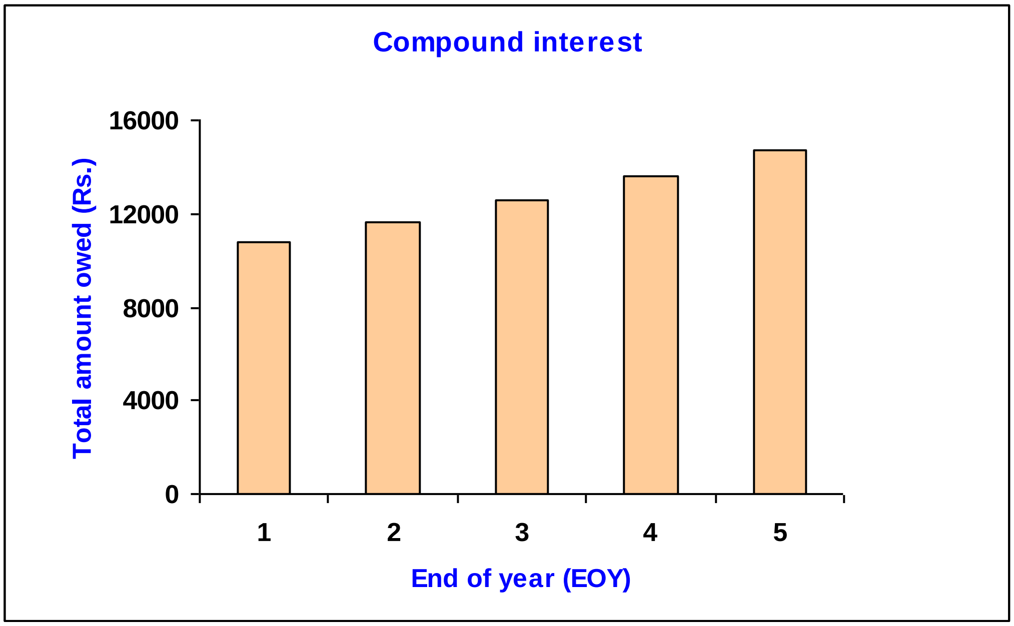


Fig. 1.2 Total amount owed (using compound interest)

The accrued interest at the end of each year considering both simple and compound interest is also shown in Fig. 1.3.

From Fig. 1.3, one can see that in case of simple interest, the amount of interest accumulated each year is constant. However in case of compound interest, the interest amount accumulated at the end of each year is not constant and increases with interest period as evident from Fig. 1.3. Thus considering compound interest, the total amount to be repaid at the end of year 5 is Rs.14,693.28, which is greater as compared to Rs.14,000 that is to be repaid on the basis of simple interest.

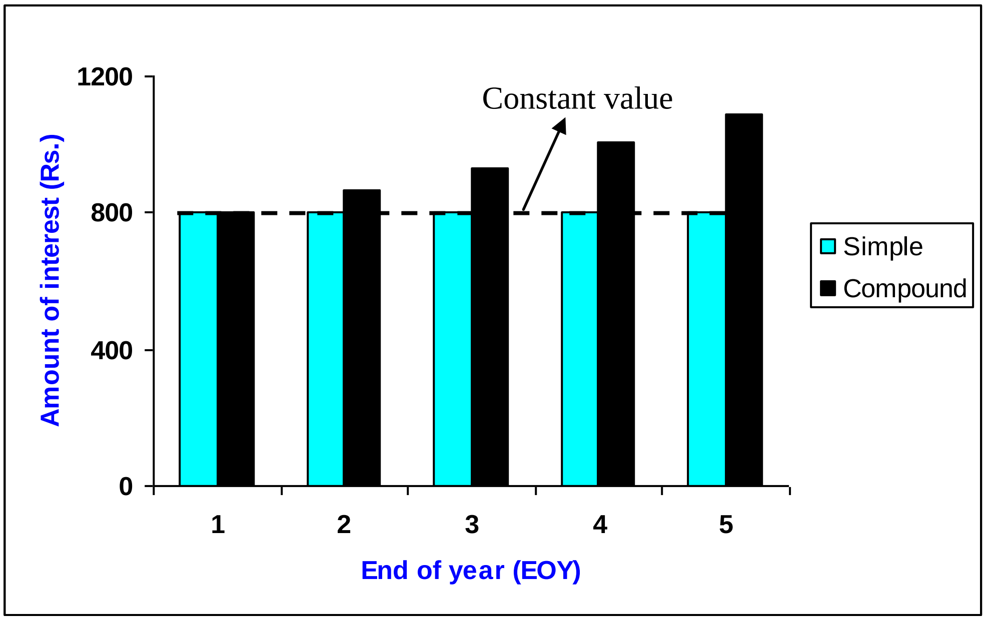


Fig. 1.3 Accumulated interest at the end of each year

Lecture-2

Equivalence:

Equivalence indicates that different amount of money at different time periods are equivalent by considering the time value of money. The following simple example will explain the meaning of equivalence.

Example: 3

What are the equivalent amounts of Rs.10000 (today) at an interest rate of 10% per year for the following cases?

a) 1 year from now (future)

b) 1 year before

Solution:

a) At interest rate of 10% per year, Rs.10000 (now) will be equivalent to Rs.11000 one year from now as shown below;

$$\text{Amount accumulated at the end of one year} = \text{Rs.}(10000 \times 1.10) = \text{Rs.}11000$$

b) Similarly Rs.10000 now was equivalent to Rs.9090.90 one year ago at interest rate of 10% per year.

$$\text{Amount one year before} = \text{Rs.}\left(10000/1.10\right) = \text{Rs.}9090.90$$

Thus due to the effect of time value of money, these amounts Rs.9090.90 (one year before), Rs.10000 (today) and Rs.11000 (one year from now) are equivalent at the interest rate of 10% per year. It is shown in Fig. 1.4.

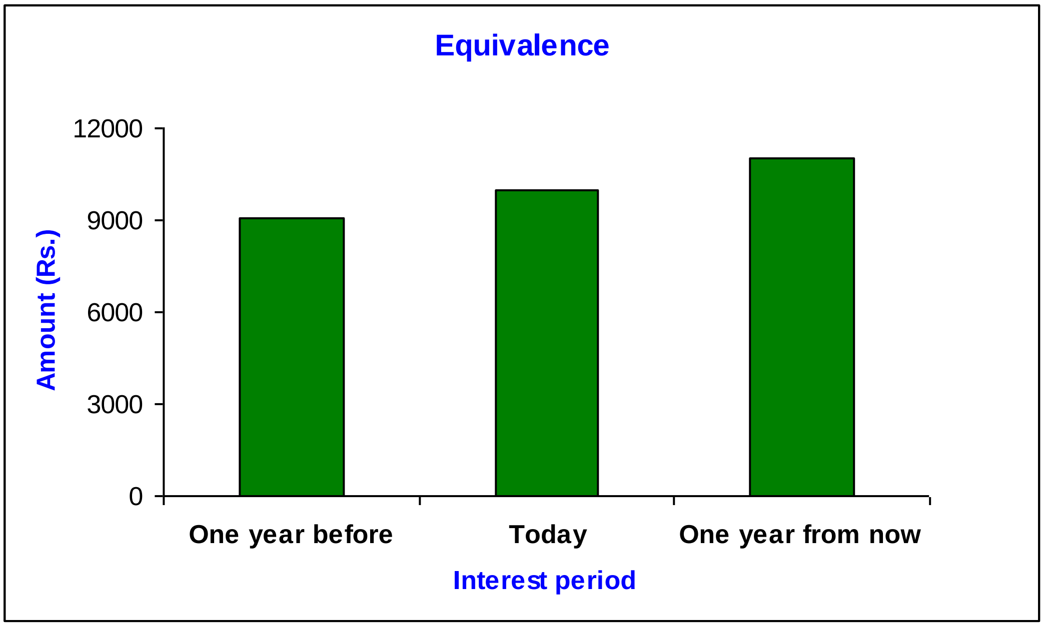


Fig 1.4 Schematic representation of equivalence

The equivalent value of an amount that is borrowed now, at future time period at a given interest rate depends on the type of interest whether simple or compound and the different loan repayment arrangements like payment of accumulated interest annually and principal at the end of the stipulated interest periods or payment of both the principal and interest at the end interest periods or payment of uniform amounts annually that comprises a portion towards the payment of principal amount and remaining for the accumulated interest throughout the interest periods.

Quantifying alternatives for decision making:

Quantifying alternatives for any item is the most important aspect of decision making for selecting the best option. For example, a construction company is planning to purchase a new concrete mixer for preparing concrete at a construction site. Let's say there are two alternatives available for purchasing the mixer; a) an automatic concrete mixer and b) a semi-automatic concrete mixer. Then the task is to find out best alternative that the company will purchase that will yield more profit. For this purpose one has to quantify both the alternatives by the following parameters;

- ✓ The initial cost that includes purchase price, sales tax, cost of delivery and cost of assembly and installation.

- ✓ Annual operating cost.
- ✓ Annual profit which will depend on the productivity i.e. quantity of concrete prepared.
- ✓ The expected useful life.
- ✓ The expected salvage value.
- ✓ Other expenditure or income (if any) associated with the equipment.
- ✓ Income tax benefit

Then on the basis of the economic criteria, the best alternative is selected by calculating the present worth or future worth or the equivalent uniform annual worth of both alternatives by incorporating the appropriate interest rate per year and the number of years (i.e. the comparison must be made over same number of years for both alternatives). Then the concrete mixer with least cost or higher net income is considered for purchase. In addition to economic parameters as mentioned above, the non-economic parameters namely environmental, social, and legal and the related regulatory and permitting process must also be considered for the evaluation and selection of the best alternative. These non-economic parameters are essentially required (in addition to the economic factors) for the selection of the best alternative for the infrastructure and heavy construction projects like dams, bridges, roadways etc. and other publicly and privately funded projects namely office buildings, hospitals, apartment building and shopping malls etc. When the available alternatives exhibit the same equivalent cost or same net income, then the non-economic parameters may play a vital role in the selection of the best alternative. It may be noted here that the non-economic parameters cannot be expressed in numerical values.

Lecture-3

Cash flow diagram:

The graphical representation of the cash flows i.e. both cash outflows and cash inflows with respect to a time scale is generally referred as cash flow diagram.

A typical cash flow diagram is shown in Fig. 1.5. The cash flows are generally indicated by vertical arrows on the time scale as shown in Fig. 1.5.

The cash outflows (i.e. costs or expense) are generally represented by vertically downward arrows whereas the cash inflows (i.e. revenue or income) are represented by vertically upward arrows.

In the cash flow diagram, number of interest periods is shown on the time scale. The interest period may be a quarter, a month or a year. Since the cash flows generally occur at different time intervals within an interest period, for ease of calculation, all the cash flows are assumed to occur at the end of an interest period. Thus in Fig. 1.5, the numbers on the time scale represent the end of year (EOY).

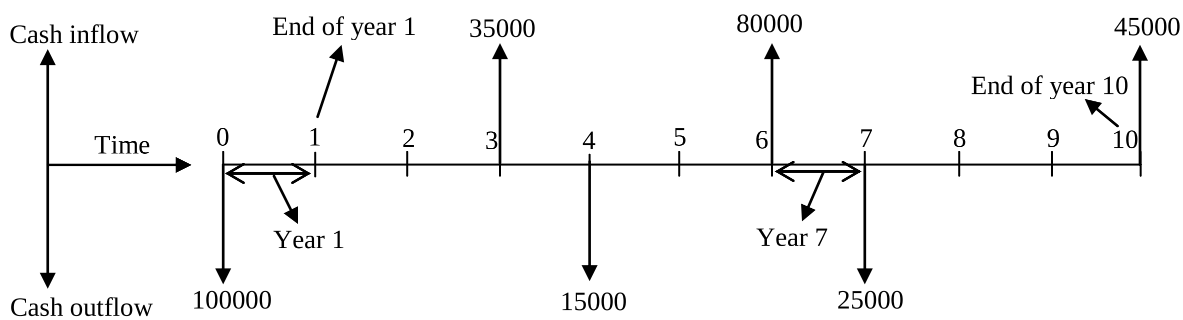


Fig 1.5 Cash flow diagram

In Fig. 1.5 the cash outflows are Rs.100000, Rs.15000 and Rs.25000 occurring at end of year (EOY) ‘0’ i.e. at the beginning, EOY ‘4’ and EOY ‘7’ respectively. Similarly the cash inflows Rs.35000, Rs.80000 and Rs.45000 are occurring at EOY ‘3’, EOY ‘6’ and EOY ‘10’ respectively.

Compound interest factors:

The compound interest factors and the corresponding formulas are used to find out the unknown amounts at a given interest rate continued for certain interest periods from the known values of varying cash flows. The following are the notations used for deriving the compound interest factors.

P = Present worth or present value

F = Future worth or future sum

A = Uniform annual worth or equivalent uniform annual worth of a uniform series continuing over a specified number of interest periods

n = number of interest periods (years or months)

i = rate of interest per interest period i.e. % per year or % per month

Unless otherwise stated, ***the rate of interest is compound interest*** and is for the entire number of interest periods i.e. for 'n' interest periods.

The present worth (P), future worth (F) and uniform annual worth (A) are shown in Fig. 1.6.

In this figure the present worth, P is at the beginning and the uniform annual series with annual value 'A' is from end of year 1 till end of year 5. Both 'P' and 'A' are cash outflows. It may be noted that the uniform annual series with annual value 'A' may be also continued throughout the entire interest periods i.e. from beginning till end of year 10 or for some intermediate interest periods like commencing from end of year 3 till end of year 8.

The future worth 'F' is occurring at end of year 4 (cash outflow), at end of year 6 (cash inflow) and at the end of year 10 (cash inflow).

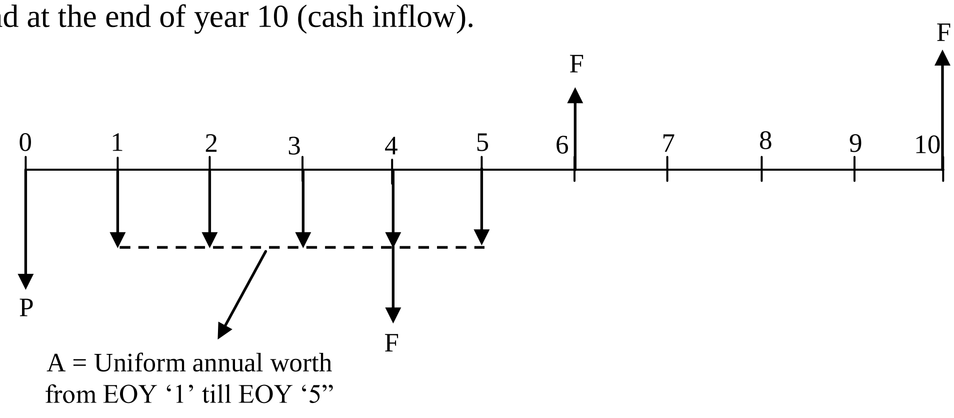


Fig 1.6 Cash flow diagram showing P, F and A

While deriving the different compound interest factors, it is assumed that the interest is ***compounded once per interest period i.e. discrete compounding***. Further the cash flows are assumed to be discrete i.e. ***they occur at the end of interest period.***

Single payment compound amount factor (SPCAF)

The single payment compound amount factor is used to compute the future worth (F) accumulated after "n" years from the known present worth (P) at a given interest rate *'i'* per interest period. It is assumed that the interest period is in years and the interest is compounded once per interest period.

The known present worth (P), unknown future worth (F) and the total interest period 'n' years are shown in Fig. 1.7.

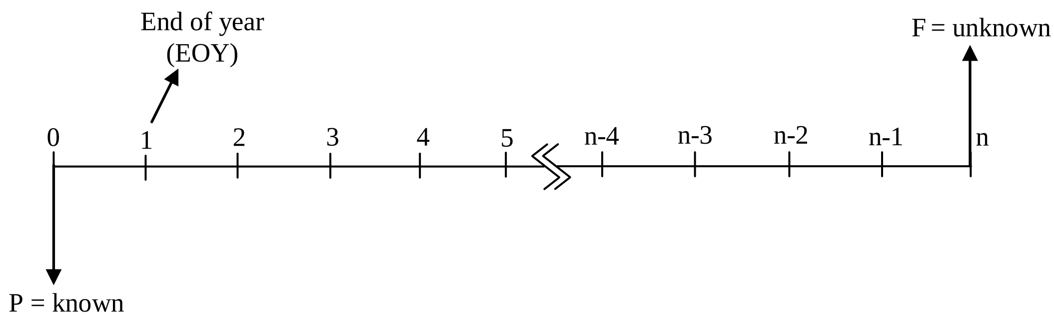


Fig 1.7 Cash flow diagram for 'known P' and 'unknown F'

The future worth (F_1) accumulated at the end of year 1 i.e. 1st year is given by;

$$F_1 = P + P \times i = P(1+i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

The future worth accumulated at the end of year 2 i.e. F_2 will be equal to the amount that was accumulated at the end of 1st year i.e. F_1 plus the amount of interest accumulated from end of 1st year to the end of 2nd year on F_1 and is given by;

$$F_2 = F_1 + F_1 \times i = F_1(1+i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Putting the value of F_1 from equation (1) in equation (2), the value of F_2 is given by;

$$F_2 = F_1(1+i) = P(1+i)(1+i) = P(1+i)^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Similarly, the future worth accumulated at the end of year 3 i.e. F_3 is equal to the amount that was accumulated at the end of 2nd year i.e. F_2 plus the amount of interest accumulated from end of 2nd year to the end of 3rd year on F_2 and is given by;

$$F_3 = F_2 + F_2 \times i = F_2(1+i) \qquad \ldots\ldots\ldots\ldots (4)$$

Putting the value of F_2 from equation (3) in equation (4), the value of F_3 is given by;

$$F_3 = F_2(1+i) = P(1+i)^2(1+i) = P(1+i)^3 \quad \ldots\ldots\ldots\ldots (5)$$

Similarly, the future worth accumulated at the end of year 4 i.e. F_4 is given by;

$$F_4 = F_3(1+i) = P(1+i)^3(1+i) = P(1+i)^4 \quad \ldots\ldots\ldots\ldots (6)$$

Thus the generalized formula for the future worth at the end of 'n' years is given by;

$$F = P(1+i)^n \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

The factor $(1+i)^n$ in equation (7) is known as the ***single payment compound amount factor (SPCAF)***. Thus if the value of 'P' is known, one can easily calculate the future worth F at the end of 'n' years at interest rate of 'i' (per year) by multiplying the present worth with the single payment compound amount factor.

Lecture-4

Single payment present worth factor (SPPWF):

The single payment present worth factor is used to determine the present worth of a known future worth (F) at the end of "n" years at a given interest rate *'i'* per interest period.

The present worth (P), future worth (F) and the total interest period 'n' years are shown in Fig. 1.8.

From equation (7), the expression for the present worth (P) can be written as follows;

$$P = F\left[\frac{1}{(1+i)^n}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

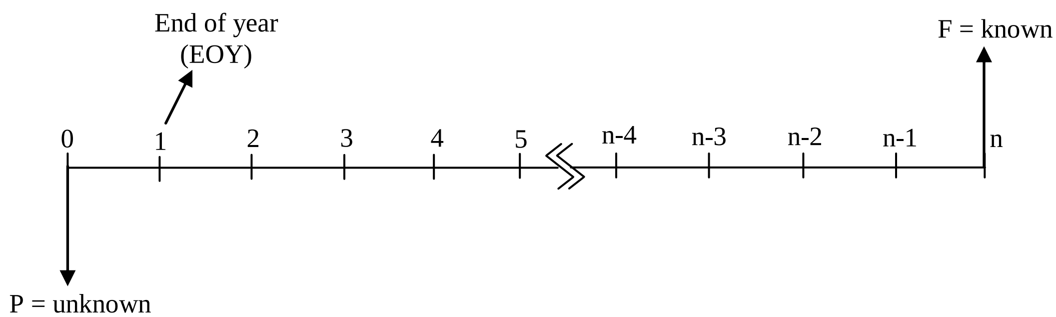


Fig 1.8 Cash flow diagram for 'known F' and 'unknown P'

The factor $1/(1+i)^n$ in equation (8) is known as ***single payment present worth factor (SPPWF).*** Thus if future worth (F) at the end of 'n' years is known, the present worth (P) at interest rate of 'i' (per year) can be calculated by multiplying the future worth with the single payment present worth factor.

Uniform series present worth factor (USPWF):

The uniform-series present worth factor is used to determine the present worth of a known uniform series. Let 'A' be the uniform annual amount at the end of each year, beginning from end of year '1' till end of year 'n'.

The known 'A', unknown 'P', and the total interest period 'n' years are shown in Fig. 1.9. This cash flow diagram refers to the case; if a person wants to get the known uniform amount of return every year, how much he has to invest now.

The present worth (P) of the uniform series can be calculated by considering each 'A' of the uniform series as the future worth. Then by using the formula in equation (8), the present worth of these future worth can be calculated and finally taking the sum of these present worth values.

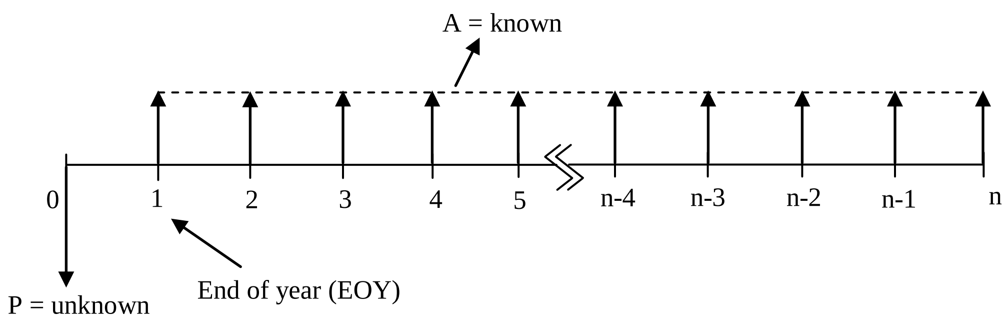


Fig 1.9 Cash flow diagram for 'known A' and 'unknown P'

The present worth (P) of the uniform series is given by;

$$P = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \cdots\cdots\cdots + \frac{A}{(1+i)^{n-2}} + \frac{A}{(1+i)^{n-1}} + \frac{A}{(1+i)^n}$$

………… (9)

$$P = A\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots\cdots\cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

…….. (10)

$$P = A\left\{(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + (1+i)^{-4} + \cdots\cdots\cdots + (1+i)^{-(n-2)} + (1+i)^{-(n-1)} + (1+i)^{-n}\right\}$$

.. (11)

The expression in the bracket is a geometric sequence with first term equal to $(1+i)^{-1}$ and common ratio equal to $(1+i)^{-1}$. Then the present worth (P) is calculated by taking the sum of the first 'n' terms of the geometric sequence (at $i \neq 0$) and is given by;

$$P = A\left[(1+i)^{-1} \times \frac{1-(1+i)^{-n}}{1-(1+i)^{-1}}\right] \quad \text{……………………..} \quad (12)$$

The simplification of equation (12) results in the following the expression;

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

The factor within the bracket in equation (13) is known as *uniform* ***series present worth factor (USPWF).*** Thus if the value of 'A' in the uniform series is known, then the present worth P at interest rate of 'i' (per year) can be calculated by multiplying the uniform annual amount 'A' with uniform series compound amount factor.

The present worth (P) of a uniform annual series of known 'A' can also be calculated in the following manner;

Dividing both sides of equation (10) by (1+i) results in the following equation;

$$\frac{P}{(1+i)} = A\left[\frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \frac{1}{(1+i)^5} + \cdots\cdots\cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} + \frac{1}{(1+i)^{n+1}}\right]$$

… (14)

Subtracting equation (14) from equation (10) results in the following expression;

$$P - \frac{P}{(1+i)} = A\left[\frac{1}{(1+i)} - \frac{1}{(1+i)^{n+1}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Equation (14) can be rewritten as;

$$\frac{P \times i}{(1+i)} = A\left[\frac{1}{(1+i)} - \frac{1}{(1+i)^{n+1}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Multiplying both sides of equation (15) by (1+i)/i and further simplification results in the following expression for the present worth 'P';

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

Capital recovery factor (CRF):

The capital recovery factor is generally used to find out the uniform annual amount 'A' of a uniform series from the known present worth at a given interest rate *'i'* per interest period.

The cash flow diagram is shown in Fig. 1.10. This cash flow diagram indicates, if a person invests a certain amount now, how much he will get as return by an equal amount each year.

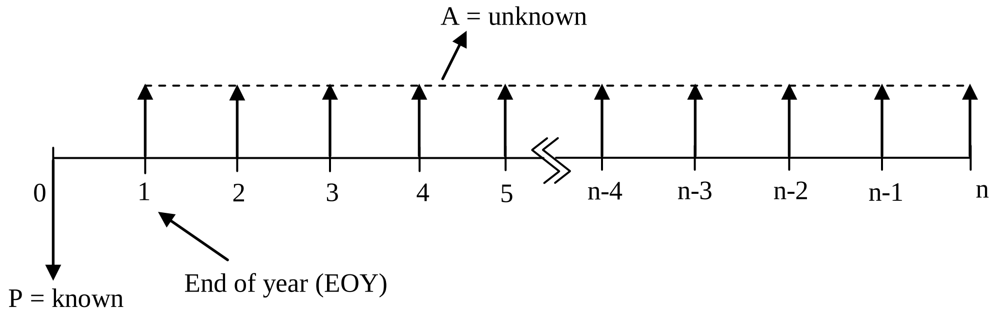


Fig 1.10 Cash flow diagram for 'known P' and 'unknown A'

From equation (12), the expression for the uniform annual amount (A) can be written as follows;

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

The factor within bracket in equation (17) is known as the ***capital recovery factor (CRF).*** Thus the uniform annual amount 'A' at interest rate of 'i' (per year) can be determined by multiplying the known present worth 'P' with the capital recovery factor.

It may be noted here that the expressions for uniform series present worth factor and capital recovery factor are derived with present worth 'P' located one interest period before the occurrence of first 'A' of the uniform series.

Lecture-5

Uniform series compound amount factor:

The uniform series compound amount factor is used to determine the future sum (F) of a known uniform annual series with uniform amount 'A'. The cash flow diagram is shown in Fig. 1.11. This cash flow diagram states that, if a person invests a uniform amount at the end of each year continued for 'n' years at interest rate of 'i' per year, how much he will get at the end of 'n' years.

Putting the value of present worth (P) from equation (8) in equation (13) results in the following;

$$F\left[\frac{1}{(1+i)^n}\right] = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (18)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

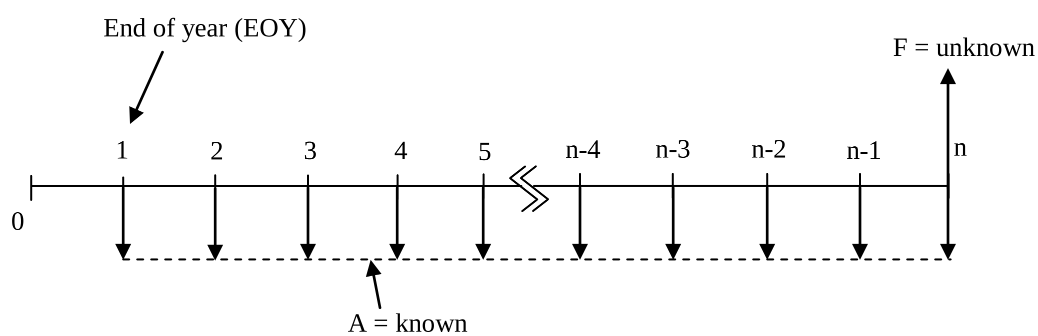


Fig 1.11 Cash flow diagram for 'known A' and 'unknown F'

The factor within bracket in equation (19) is known as ***uniform series compound amount factor (USCAF).*** Hence the future worth 'F' can be computed by multiplying the uniform annual amount 'A' with the uniform series compound amount factor.

Sinking fund factor:

The sinking fund factor is used to calculate the annual amount 'A' of a uniform series from the known future sum 'F'. The cash flow diagram is shown in Fig. 1.12. This cash flow diagram indicates that, if a person wants to get a known future sum at the end of 'n' years at interest rate of 'i' per year, how much he has to invest every year by an equal amount.

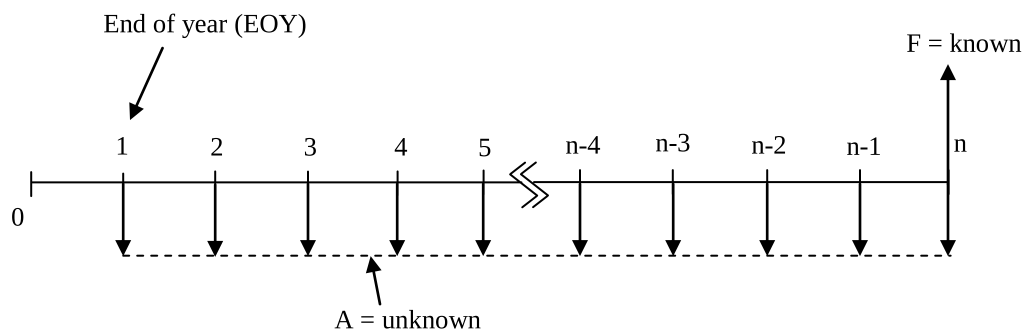


Fig 1.12 Cash flow diagram for 'known F' and 'unknown A'

From equation (19), the expression for the uniform annual amount (A) can be written as follows;

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (20)$$

The factor within bracket in equation (20) is known as ***sinking fund factor (SFF).*** Thus one can find out the annual amount 'A' of a uniform series by multiplying the future worth 'F' with the sinking fund factor.

The derivation of expressions for both uniform series compound amount factor and sinking fund factor is based on the fact that, the future sum 'F' occurs at the same time as the last 'A' of the uniform series.

The compound interest factors with details are presented Table 1.3.

Table 1.3 Interest factors (Discrete compounding) with rate of interest '*i*' (%) and number of interest periods 'n'

Name of the factor (1)	Abbreviation (2)	Functional representation (3)	Mathematical expression (4)	Given (5)	To find out (6) = (4) x (5)
Single payment compound amount factor	*SPCAF*	$(F/P,i,n)$	$(1+i)^n$	P	F
Single payment present worth factor	*SPPWF*	$(P/F,i,n)$	$\frac{1}{(1+i)^n}$	F	P
Uniform series present worth factor	*USPWF*	$(P/A,i,n)$	$\frac{(1+i)^n-1}{i(1+i)^n}$	A	P
Capital recovery factor	*CRF*	$(A/P,i,n)$	$\frac{i(1+i)^n}{(1+i)^n-1}$	P	A
Uniform series compound amount factor	*USCAF*	$(F/A,i,n)$	$\frac{(1+i)^n-1}{i}$	A	F
Sinking fund factor	*SFF*	$(A/F,i,n)$	$\frac{i}{(1+i)^n-1}$	F	A

Using this information, one can easily find out the required expenditure or income i.e. may be 'P', 'F' or 'A' from the know cash flows. For example if it is required to find out the uniform annual amount 'A' of a uniform series from the know value of present worth 'P', then multiply the capital recovery factor (CRF) i.e. $\frac{i(1+i)^n}{(1+i)^n-1}$ with 'P' to get the required uniform annual amount 'A' (highlighted in the above table).

In addition, the use of functional representations of the compound interest factors presented in this table can be used to find out the unknown amounts. For illustration, if it is required to find out 'A' from known 'P', then expression for 'A' can be written as follows;

$$A = P\left(\frac{A}{P},i,n\right)$$

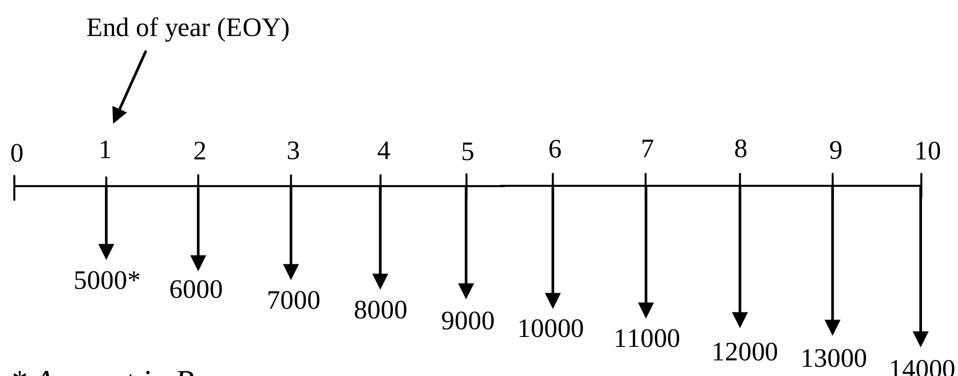
Further the functional representation of a compound interest factor can be obtained by multiplying the other relevant compound interest factors at a given rate of interest 'i' and number of interest periods 'n' and is given as follows;

$$\left(\frac{A}{P},i,n\right)=\left(\frac{A}{F},i,n\right)\times\left(\frac{F}{P},i,n\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(21)}$$

Lecture-6

Cash flow involving arithmetic gradient payments or receipts:

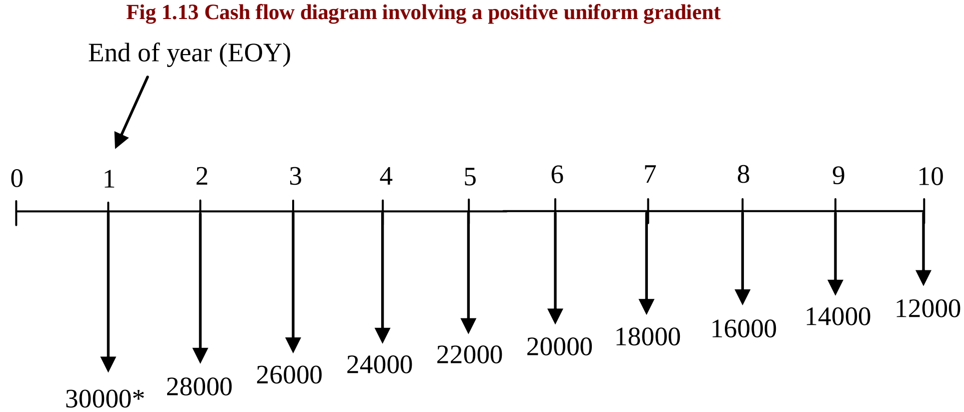
Some cash flows involve the payments or receipts in gradients by same amount. In other words, the expenditure or the income increases or decreases by same amount. The cash flow involving such payments or receipts is known as uniform gradient series. For example, if the cost of repair and maintenance of a piece of equipment increases by same amount every year till end of its useful life, it represents a cash flow involving positive uniform gradient. Similarly if the profit obtained from an investment decreases by an equal amount every year for a certain number of years, it indicates a cash flow involving negative uniform gradient. The cash flow diagrams for positive gradient and negative gradient are shown in Fig. 1.3 and Fig. 1.4 respectively.



** Amount in Rupees.*

A positive gradient of Rs.1000.

Fig 1.13 Cash flow diagram involving a positive uniform gradient



** Amount in Rupees.*

A negative gradient of Rs.2000.

Fig 1.14 Cash flow diagram involving a negative uniform gradient

The present worth, future worth and the equivalent uniform annual worth of the uniform gradient can be derived using the compound interest factors. The generalized cash flow diagram involving a positive uniform gradient with base value 'B' and the gradient 'G' is shown in Fig. 1.15a. The cash flow shown in Fig. 1.15a can be split into two cash flows; one having the uniform series with amount 'B' and the other having the gradient series with values in multiples of gradient amount 'G' and is shown in Fig. 1.15b. This gradient series is also know as the arithmetic gradient series as the expense or the income increases by the uniform arithmetic amount 'G' every year.

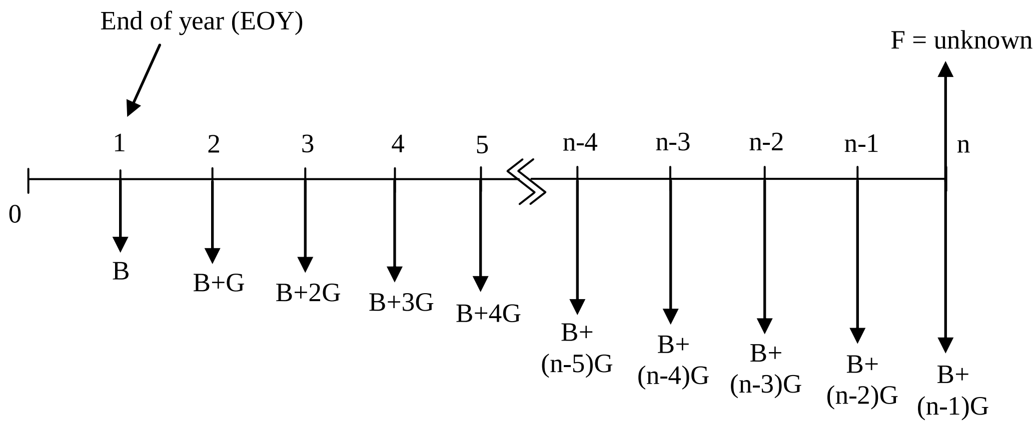


Fig 1.15a Generalized Cash flow diagram involving a positive uniform gradient with 'unknown F'

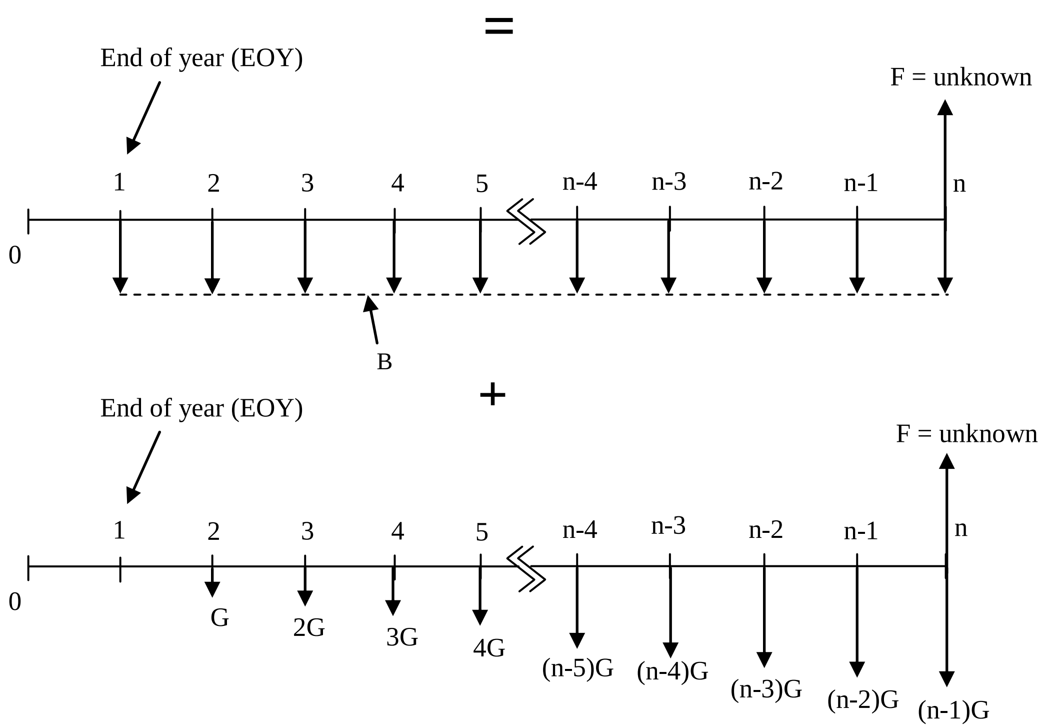


Fig 1.15b Cash flow diagram (positive uniform gradient) for 'unknown F'

The future worth of the cash flow shown in Fig. 1.15a can be obtained by finding the out the individual future worth of the cash flows shown in Fig. 1.15b and then taking their sum. As already stated, the future worth of the cash flow involving an uniform series can be determined by multiplying the uniform annual amount ‘B’ with the uniform series compound amount factor. The future worth (F) of the gradient series shown in Fig 1.15b can be determined by finding out the individual future worth of the gradient values (i.e. in multiples of gradient amount ‘G’) at the end of different years at interest rate of ‘i’ per year and then taking the sum of these individual futures values. Then ‘F’ is given as follows;

$$F=\left\{\begin{array}{l} G(1+i)^{n-2}+2G(1+i)^{n-3}+3G(1+i)^{n-4}+4G(1+i)^{n-5}+\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots+(n-4)G(1+i)^{3}+(n-3)G(1+i)^{2}+(n-2)G(1+i)+(n-1)G \end{array}\right\}\ldots\ldots$$

(22)

Dividing both sides of equation (22) by $(1+i)^n$ results in the following equation;

$$\frac{F}{(1+i)^n}=G\left\{\begin{array}{l} \frac{1}{(1+i)^2}+\frac{2}{(1+i)^3}+\frac{3}{(1+i)^4}+\frac{4}{(1+i)^5}+\cdots\cdots\cdots\cdots\cdots \\ \qquad\cdots\cdots\cdots+\frac{n-4}{(1+i)^{n-3}}+\frac{n-3}{(1+i)^{n-2}}+\frac{n-2}{(1+i)^{n-1}}+\frac{n-1}{(1+i)^{n}} \end{array}\right\}\cdots$$

…. (23)

Multiplying both sides of equation (23) by (1+i) results in the following expression;

$$\frac{F(1+i)}{(1+i)^n}=G\left\{\begin{array}{l} \frac{1}{(1+i)}+\frac{2}{(1+i)^2}+\frac{3}{(1+i)^3}+\frac{4}{(1+i)^4}+\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\cdots\cdots\cdots+\frac{n-4}{(1+i)^{n-4}}+\frac{n-3}{(1+i)^{n-3}}+\frac{n-2}{(1+i)^{n-2}}+\frac{n-1}{(1+i)^{n-1}} \end{array}\right\}$$

……... (24)

Subtracting equation (23) from equation (24) results in the following expression;

$$F\left[\frac{(1+i)-1}{(1+i)^n}\right]=G\left\{\begin{array}{l} \frac{1}{(1+i)}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\frac{1}{(1+i)^4}+\cdots\cdots\cdots\cdots\cdots\cdots \\ \quad\cdots\cdots+\frac{1}{(1+i)^{n-4}}+\frac{1}{(1+i)^{n-3}}+\frac{1}{(1+i)^{n-2}}+\frac{1}{(1+i)^{n-1}}+\frac{1}{(1+i)^{n}}-\frac{n}{(1+i)^{n}} \end{array}\right\}$$

... (25)

The equation (25) can be rewritten as follows;

$$F\left[\frac{i}{(1+i)^n}\right] = G\left\{\begin{array}{l}\dfrac{1}{(1+i)}+\dfrac{1}{(1+i)^2}+\dfrac{1}{(1+i)^3}+\dfrac{1}{(1+i)^4}+\cdots\cdots\cdots \\ \cdots\cdots+\dfrac{1}{(1+i)^{n-4}}+\dfrac{1}{(1+i)^{n-3}}+\dfrac{1}{(1+i)^{n-2}}+\dfrac{1}{(1+i)^{n-1}}+\dfrac{1}{(1+i)^n}\end{array}\right\}-\frac{nG}{(1+i)^n}$$

….. (26)

The expression in the bracket is a geometric sequence with first term equal to 1/(1+i) and common ratio equal to 1/(1+i). Thus equation (26) is rewritten by taking sum of the first 'n' terms of the geometric sequence (at $i \neq 0$) and is given by;

$$F\left[\frac{i}{(1+i)^n}\right] = G\left[\frac{1}{(1+i)}\times\frac{1-\dfrac{1}{(1+i)^n}}{1-\dfrac{1}{(1+i)}}\right]-\frac{nG}{(1+i)^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (27)$$

The expression for future worth 'F' can be written as follows:

$$F = G\left[\frac{(1+i)^n-1}{i^2}-\frac{n}{i}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (28)$$

The expression in the bracket is known as the uniform gradient future worth factor (UGFWF). With functional representation equation (28) can be rewritten as follows;

$$F = G\left(\frac{F}{G}, i, n\right)$$

Thus the functional representation of uniform gradient future worth factor (UGFWF) can be written as follows;

$$\left(\frac{F}{G}, i, n\right) = \left[\frac{(1+i)^n-1}{i^2}-\frac{n}{i}\right] \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

It may be noted here that the future worth 'F' occurs in the same year as that of the last gradient amount. Further the functional representations of other gradient factors can be found out by multiplying the relevant gradient factor with the compound interest factor at the given rate of interest 'i' per interest period and number of interest periods 'n'.

Lecture-7

Arithmetic gradient:

The present worth factor of a uniform gradient series (i.e. arithmetic gradient series) with values in multiples of gradient amount 'G' can be obtained by multiplying the uniform gradient future worth factor (UGFWF) with the single payment present worth factor (SPPWF) and the functional representation is given as follows

$$\left(\frac{P}{G}, i, n\right) = \left(\frac{F}{G}, i, n\right) \times \left(\frac{P}{F}, i, n\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(30)}$$

$\left(\frac{P}{G}, i, n\right)$ is known as the uniform gradient present worth factor (UGPWF).

Now putting the expressions for uniform gradient future worth factor (UGFWF) and single payment present worth factor (SPPWF) in equation (30) results in the following expressions;

$$\left(\frac{P}{G}, i, n\right) = \left[\frac{(1+i)^n - 1}{i^2} - \frac{n}{i}\right] \times \left[\frac{1}{(1+i)^n}\right] \quad \ldots\ldots\ldots\ldots \text{(31)}$$

On further simplification results in the following equation;

$$\left(\frac{P}{G}, i, n\right) = \frac{1}{i(1+i)^n}\left[\frac{(1+i)^n - 1}{i} - n\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(32)}$$

Thus the expression of the present worth 'P' can be written as follows

$$P = G\left(\frac{P}{G}, i, n\right)$$

The cash flow diagram is shown in Fig. 1.16. It may be noted here that the gradient starts at the end of year '2' whereas the present worth occurs at the beginning i.e. in year 0.

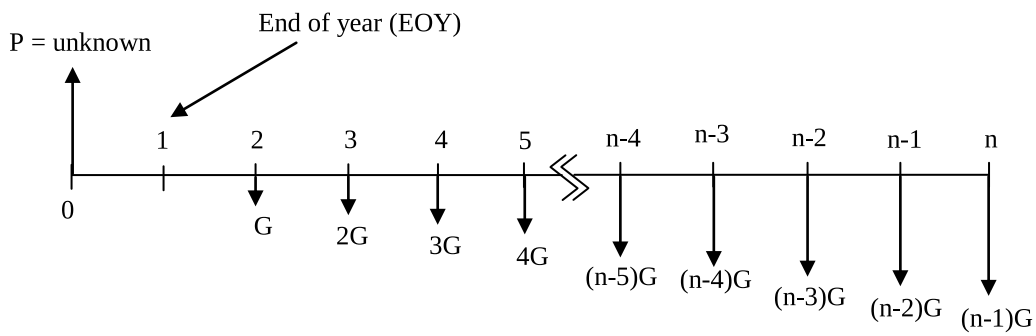


Fig. 1.16 Cash flow diagram involving a uniform gradient with 'unknown P'

Similarly the annual worth factor of a uniform gradient series can be obtained by multiplying the uniform gradient future worth factor (UGFWF) with the sinking fund factor (SFF) and the functional representation is presented as follows;

$$\left(\frac{A}{G},i,n\right)=\left(\frac{F}{G},i,n\right)\times\left(\frac{A}{F},i,n\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

$\left(\frac{A}{G},i,n\right)$ is known as the uniform gradient annual worth factor (UGAWF).

Thus the expression for the uniform gradient annual worth factor is obtained by putting the expressions for uniform gradient future worth factor (UGFWF) and sinking fund factor (SFF) in equation (33) and is given as follows;

$$\left(\frac{A}{G},i,n\right)=\left[\frac{(1+i)^n-1}{i^2}-\frac{n}{i}\right]\times\left[\frac{i}{(1+i)^n-1}\right] \quad \ldots\ldots\ldots\ldots (34)$$

$$\left(\frac{A}{G},i,n\right)=\left[\frac{1}{i}-\frac{n}{(1+i)^n-1}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

The expression of the equivalent uniform annual worth 'A' of the uniform gradient series can be written as follows;

$$A=G\left(\frac{A}{G},i,n\right)$$

The cash flow diagram is shown in Fig. 1.17. It may be noted here that, the equivalent uniform annual worth 'A' occurs from end of year '1' to end of year 'n' whereas the gradient starts at the end of year '2'.

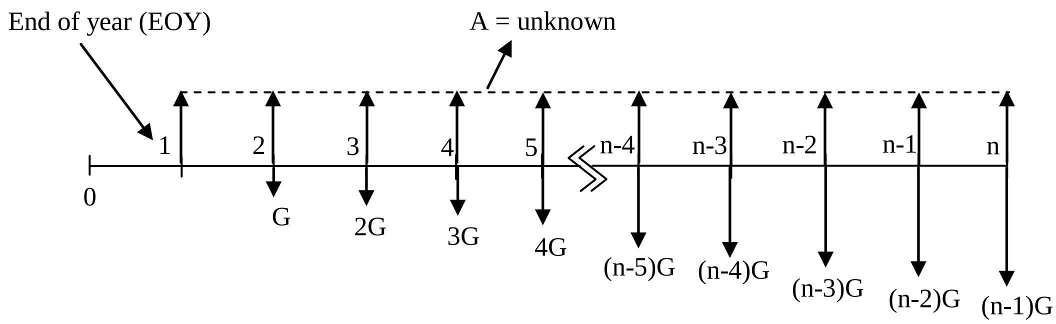


Fig. 1.17 Cash flow diagram involving a uniform gradient with 'unknown A'

Now one cane easily find out the present worth, future worth and equivalent uniform annual worth of the cash flow shown in Fig. 1.15a (a positive uniform gradient with base value 'B' and the gradient 'G') by taking the sum of the corresponding values from the uniform series with uniform annual amount 'B' i.e. base value and the gradient series with values in multiples of gradient amount 'G'. Similarly the present worth, future worth and equivalent uniform annual worth of a cash flow involving a negative uniform gradient can be calculated by subtracting the corresponding values of the gradient series (with values in multiples of gradient amount 'G') from the uniform series with uniform annual amount 'B' i.e. base value as shown Fig. 1.18a and Fig. 1.18b.

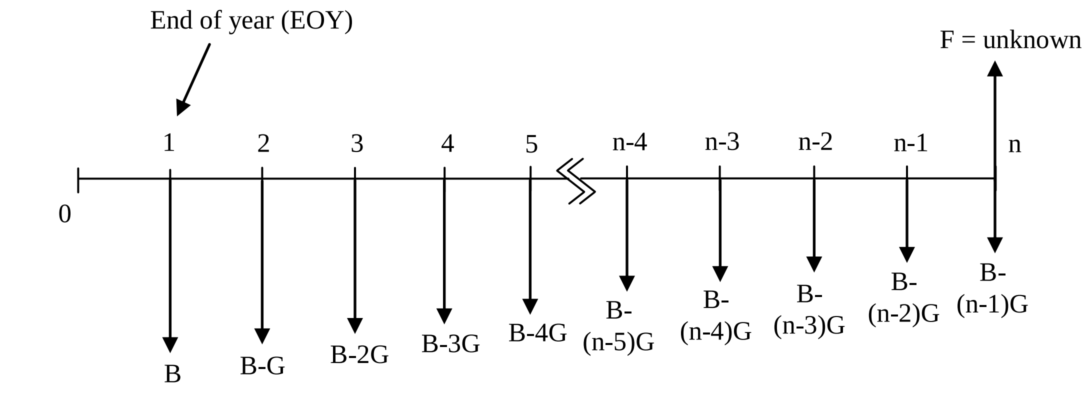


Fig 1.18a Generalized Cash flow diagram involving a negative uniform gradient with 'unknown F'

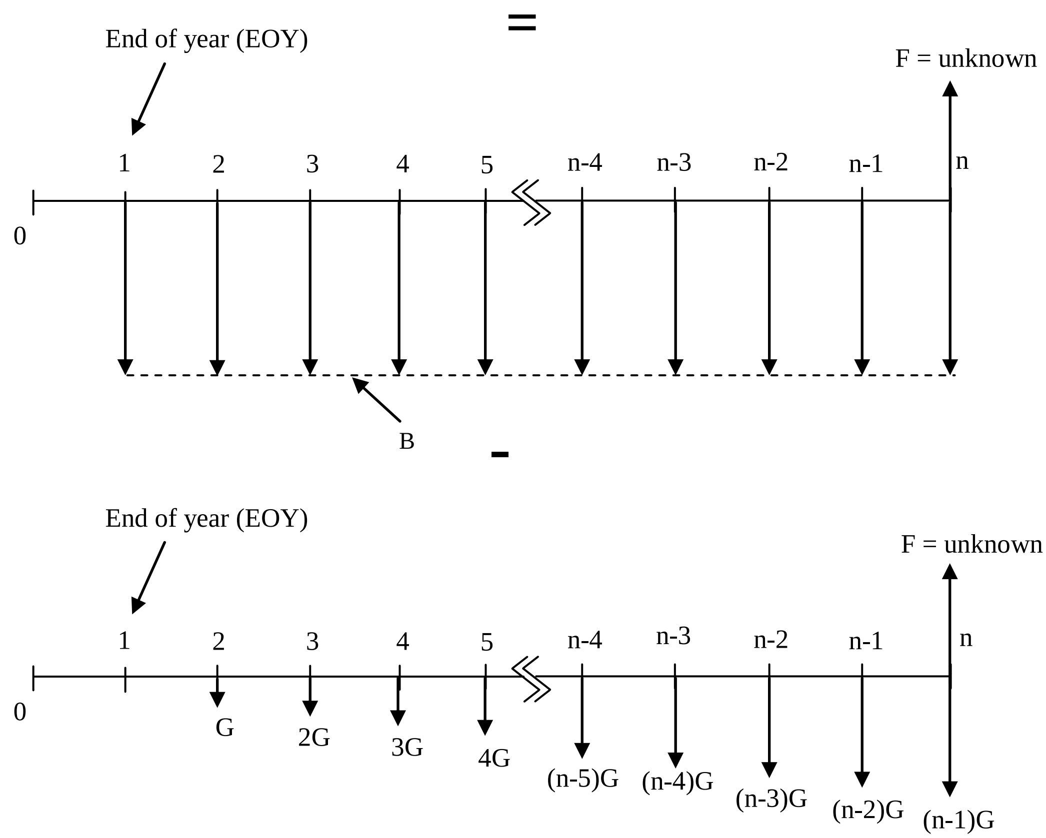


Fig 1.18b Cash flow diagram (negative uniform gradient) for 'unknown F'

Lecture-8

Cash flow involving geometric gradient series:

Sometimes the cash flows may have expenses or incomes being increased by a constant percentage in the successive time periods i.e. in successive years. Such kind of cash flow is known as geometric gradient series. The generalized cash flow diagram involving geometric gradient series with expense or receipt 'C' at the end of year '1" and geometric percentage increase 'g' is shown in Fig. 1.19.

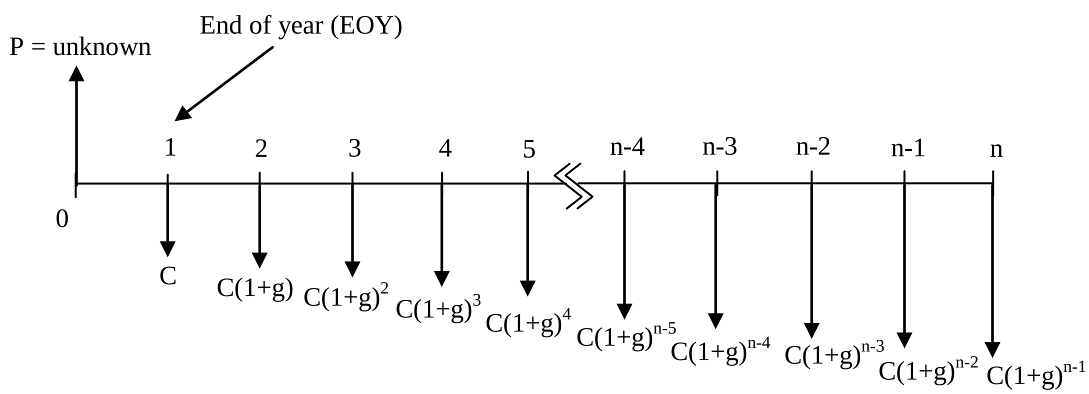


Fig 1.19 Cash flow diagram involving geometric gradient with 'unknown P'

The present worth (P) of the geometric gradient series can be calculated by considering each amount as the future worth and then taking sum of these present worth values.

$$P = \left\{ \frac{C}{(1+i)} + \frac{C(1+g)}{(1+i)^2} + \frac{C(1+g)^2}{(1+i)^3} + \frac{C(1+g)^3}{(1+i)^4} + \cdots\cdots\cdots + \frac{C(1+g)^{n-3}}{(1+i)^{n-2}} + \frac{C(1+g)^{n-2}}{(1+i)^{n-1}} + \frac{C(1+g)^{n-1}}{(1+i)^n} \right\}$$

………….. (36)

Equation (36) can be rewritten as follows;

$$P = C\left\{ \frac{1}{(1+i)} + \frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \frac{(1+g)^3}{(1+i)^4} + \cdots\cdots\cdots + \frac{(1+g)^{n-3}}{(1+i)^{n-2}} + \frac{(1+g)^{n-2}}{(1+i)^{n-1}} + \frac{(1+g)^{n-1}}{(1+i)^n} \right\}$$

………….. (37)

Dividing both sides of the equation (37) by (1+i) results in the following expression;

$$\frac{P}{(1+i)} = C\left\{\frac{1}{(1+i)^2} + \frac{(1+g)}{(1+i)^3} + \frac{(1+g)^2}{(1+i)^4} + \frac{(1+g)^3}{(1+i)^5} + \cdots\cdots\cdots + \frac{(1+g)^{n-3}}{(1+i)^{n-1}} + \frac{(1+g)^{n-2}}{(1+i)^n} + \frac{(1+g)^{n-1}}{(1+i)^{n+1}}\right\}$$

…………. (38)

Now multiplying both sides of equation (38) by (1+g) results in the following expression;

$$\frac{P(1+g)}{(1+i)} = C\left\{\frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \frac{(1+g)^3}{(1+i)^4} + \frac{(1+g)^4}{(1+i)^5} + \cdots\cdots\cdots + \frac{(1+g)^{n-2}}{(1+i)^{n-1}} + \frac{(1+g)^{n-1}}{(1+i)^n} + \frac{(1+g)^n}{(1+i)^{n+1}}\right\}$$

…………. (39)

Subtracting equation (37) from equation (39) results in the following;

$$\frac{P(g-i)}{(1+i)} = C\left\{\frac{(1+g)^n}{(1+i)^{n+1}} - \frac{1}{(1+i)}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Equation (40) can be rewritten as follows;

$$P = \frac{C\left\{\frac{(1+g)^n}{(1+i)^n} - 1\right\}}{(g-i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

The expression in equation (41) is valid when g ≠ i. When 'g' is equal to 'i' then right hand side of the equation (41) takes the indeterminate form of $\frac{0}{0}$. In this case the expression for the present worth 'P' can be obtained by applying L'Hospital's rule as follows;

$$\lim_{g\to i}\frac{C\left\{\frac{(1+g)^n}{(1+i)^n} - 1\right\}}{(g-i)} = \lim_{g\to i}\frac{\frac{d\left[C\left\{\frac{(1+g)^n}{(1+i)^n} - 1\right\}\right]}{dg}}{\frac{d\left[(g-i)\right]}{dg}} = \lim_{g\to i}\frac{\frac{Cn(1+g)^{n-1}}{(1+i)^n}}{1} = \frac{Cn}{(1+i)} \quad \ldots\ldots (42)$$

Then the expression for the present worth 'P' is given as follows;

$$P = \frac{Cn}{(1+i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

Similarly the future worth and equivalent uniform annual worth of the geometric gradient series can be obtained by multiplying its present worth 'P' with the compound interest factors namely single payment compound amount factor (SPCAF) and capital recovery factor (CRF) respectively at the given rate of interest 'i' per interest period and the number of interest periods.

Now using the above expressions, one can easily calculate the present worth, future worth and equivalent uniform annual worth of the cash flow involving either expenditure or income or both increasing in the form of geometric gradient.

It may be noted here that the expressions for compound interest factors can also be obtained by using beginning of year (interest period) convention.

The values of discrete compound interest factors at different values of interest rate and interest period can be calculated by using the expressions of these factors as mentioned earlier and the interest tables can thus be generated. The interest tables are available in the relevant texts (cited in the list of references for this course). The values of different discrete compound interest factors from these interest tables can directly be used in the engineering economy calculations.

The use of different compound interest factors discussed so far will be illustrated by solving various examples for comparison of different alternatives in Module 2. Further the use of discrete compound interest factors from interest tables will also be illustrated in Module 2.