Module 4 : Deflection of Structures

Lecture 6 : Maxwell-Betti Law of Reciprocal Deflections

Objectives

In this course you will learn the following

- Maxwell-Betti Law of reciprocal deflection.
- Illustrative examples for proving law of reciprocal deflection.

4.7 Maxwell-Betti Law of Reciprocal Deflections

Maxwell-Betti Law of real work is a basic theorem in the structural analysis. Using this theorem, it will be established that the flexibility coefficients in compatibility equations, formulated to solve indeterminate structures by the flexibility method, form a symmetric matrix and this will reduce the number of deflection computations. The Maxwell-Betti law also has applications in the construction of influence lines diagrams for statically indeterminate structures. The Maxwell-Betti law, which applies to any stable elastic structure (a beam, truss, or frame, for example) on unyielding supports and at constant temperature, states:

The deflection of point *A* in direction 1 due to unit load at point *B* in direction 2 is equal in the magnitude to the deflection of point *B* in direction 2 produced by a unit load applied at *A* in direction 1.

The Figure 4.31 explains the Maxwell-Betti Law of reciprocal displacements in which, the displacement Δ_{AB} is equal to the displacement Δ_{BA}.

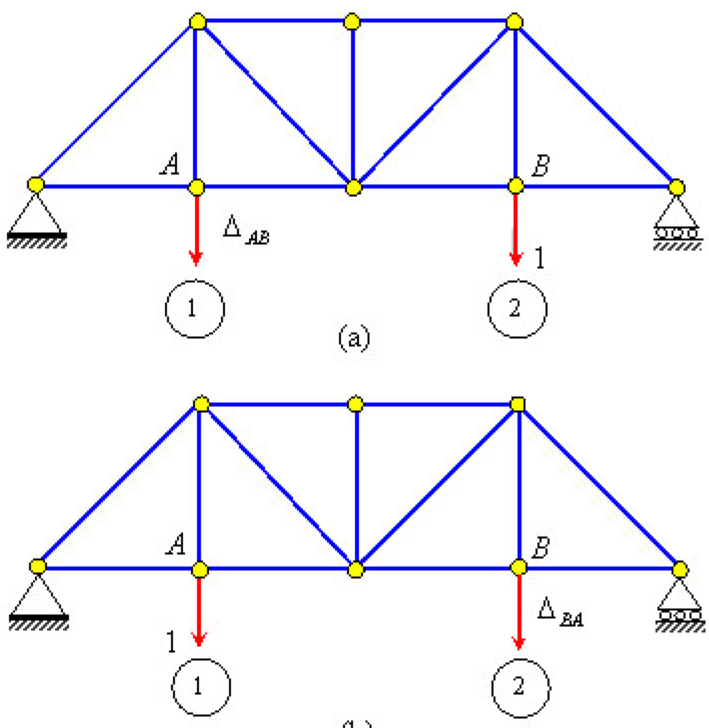
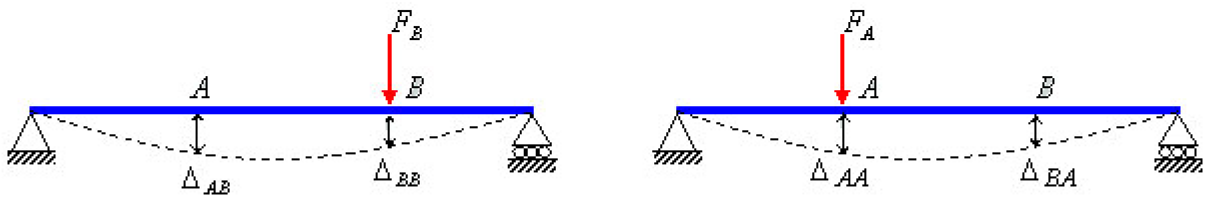


Figure 4.31 Illustration of Maxwell-Betti Law (directions 1 and 2 are shown by circle)



(a) (b)

Figure 4.32

In order to prove the reciprocal theorem, consider the simple beams shown in Figure 4.32.

Let a vertical force F_B at point *B* produces a vertical deflection Δ_{AB} at point A and Δ_{BB} at point *B* as shown in Figure 4.32(a). Similarly, in Figure 4.32(b) the application of a vertical force F_A at point *A* produces a vertical deflections Δ_{AA} and Δ_{BA} at points *A* and *B* , respectively. Let us evaluate the total work done by the two forces F_A and F_B when they are applied in different order to the zero to their final value.

Case 1: F_B applied and followed by F_A

(a) Work done when F_B is gradually applied

$$W_B = \frac{1}{2} F_B \Delta_{BB}$$

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(b) Work done when F_A is gradually applied with F_B in place

$$W_A = \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA}$$

Total work done by the two forces for case 1 is

$$W_1 = W_B + W_A$$

$$= \frac{1}{2} F_B \Delta_{BB} + \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA} \qquad \textbf{(4.29)}$$

Case2: F_A applied and followed by F_B

(c) Work done when F_A is gradually applied

$$W_A = \frac{1}{2} F_A \Delta_{AA}$$

(d) Work done when F_B is gradually applied with F_A in place

$$W_B = \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

Total work done by the two forces for case 2 is

$$W_2 = W_B + W_A$$

$$= \frac{1}{2} F_A \Delta_{AA} + \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB} \qquad \textbf{(4.30)}$$

Since the final deflected position of the beam produced by the two cases of loads is the same regardless of the order in which the loads are applied. The total work done by the forces is also the same regardless of the order in which the loads are applied. Thus, equating the total work of Cases 1 and 2 give

$$W_1 = W_2$$

$$\frac{1}{2}F_B\Delta_{BB} + \frac{1}{2}F_A\Delta_{AA} + F_B\Delta_{BA} = \frac{1}{2}F_A\Delta_{AA} + \frac{1}{2}F_B\Delta_{BB} + F_A\Delta_{AB}$$

$$F_B\Delta_{BA} = F_A\Delta_{AB} \quad \textbf{(4.31)}$$

If $F_A = F_B = 1$, the equation (4.31) depicts the statement of the Maxwell-Betti law i.e.

$$\Delta_{BA} = \Delta_{AB}$$

The Maxwell-Betti theorem also holds for rotations as well as rotation and linear displacement in beams and frames.

Example 4.21 Verify Maxwell-Betti law of reciprocal displacement for the direction 1 and 2 of the pin-jointed structure shown in Figure 4.33(a).

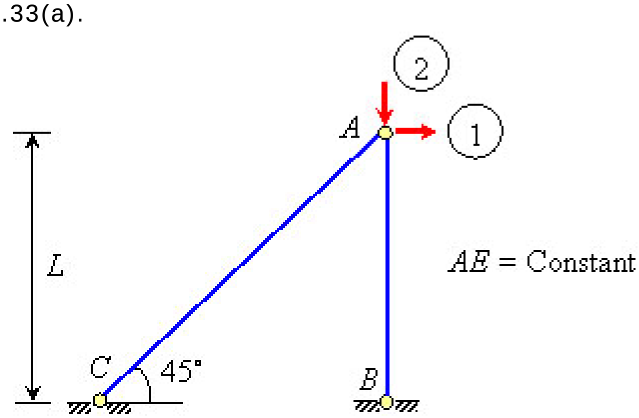

Figure 4.33(a)

Solution: Apply the forces P_1 and P_2 in the direction 1 and 2, respectively. The calculation of total strain energy in the system is given in Table 4.5.

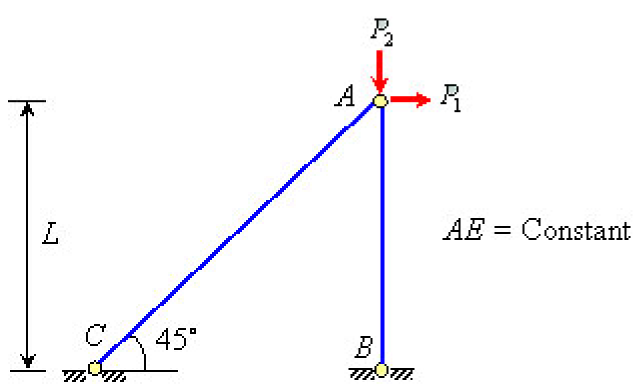

Figure 4.33(a)

Table 4.5

Member	Length	Force *P*	$U = P^2L/2AE$
AB	***L***	$-(P_1+P_2)$	$(P_1+P_2)^2 L/2AE$
AC	$\sqrt{2}$ ***L***	$\sqrt{2}$ ***P*** 1	$\sqrt{2}P_1^2 L/AE$

$$\sum \quad ((P_1+P_2)^2 + 2\sqrt{2}P_1^2)L/2AE$$

$$\Delta_{21} = \left.\frac{\partial U}{\partial P_2}\right|_{P_1=1, P_2=0}$$

$$= (2(P_1+P_2)1 + 0)\left.\frac{L}{2AE}\right|_{P_1=1, P_2=0}$$

$$= \frac{L}{AE}$$

$$\Delta_{12} = \left.\frac{\partial U}{\partial P_1}\right|_{P_1=0, P_2=1}$$

$$= (2(P_1+P_2)1 + 4\sqrt{2}P_1)\left.\frac{L}{2AE}\right|_{P_1=0, P_2=1}$$

$$= \frac{L}{AE}$$

Since $\Delta_{12} = \Delta_{21}$, hence the Maxwell-Betti law of reciprocal displacement is proved.

Example 4.22 Verify Maxwell-Betti law of reciprocal displacement for the cantilever beam shown in Figure 4.34(a).

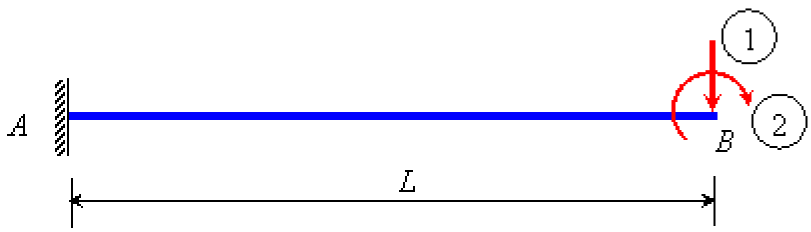

Figure 4.34(a)

Solution: Apply the forces P_1 and P_2 in the directions 1 and 2, respectively. The total strain energy stored is calculated below.

Consider any point *X* at a distance *x* from *B* .

$$M_x = -(P_1 x + P_2)$$

$$U = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_0^L (P_1 x + P_2)^2 dx$$

$$= \frac{1}{2EI}\left(\frac{P_1^2 L^3}{3} + P_1 P_2 L^2 + P_2^2 L\right)$$

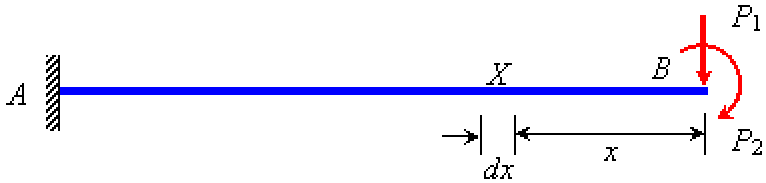

Figure 4.34(b)

$$\Delta_{12} = \left.\frac{\partial U}{\partial P_1}\right|_{P_1=0, P_2=1}$$

$$= \frac{1}{2EI}\left(\frac{2P_1L^3}{3} + P_2L^2 + 0\right)\Bigg|_{P_1=0, P_2=1}$$

$$= \frac{L^2}{2EI}$$

$$\Delta_{21} = \left.\frac{\partial U}{\partial P_2}\right|_{P_1=1, P_2=0}$$

$$= \frac{1}{2EI}\left(0 + P_1L^2 + 2P_2L\right)\Big|_{P_1=1, P_2=0}$$

$$= \frac{L^2}{2EI}$$

Since $\Delta_{12} = \Delta_{21}$, the Maxwekk-Betti law of reciprical displacement is proved.

Example 4.23 Verify Maxwell-Betti law of reciprocal displacement for the rigid-jointed plane frame with reference to marked direction as shown in Figure 4.35(a). *EI* is same for both members.

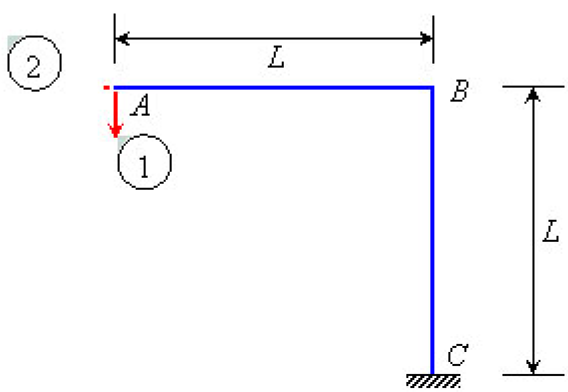

Figure 4.35(a)

Solution: Apply the forces P_1 and P_2 in the directions 1 and 2, respectively as shown in Figure 4.35(b).

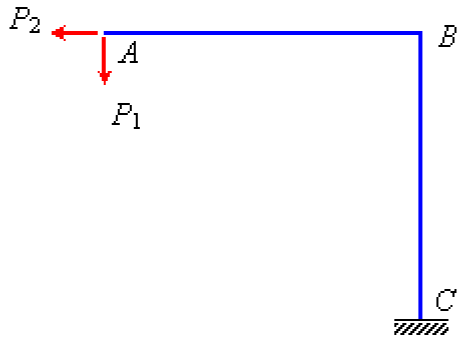

Figure 4.35(b)

Consider *AB* : (*x* measured from *A*)

$$M_x = -P_1 x$$

$$U_{AB} = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_0^L (-P_1 x)^2 dx$$

$$= \frac{P_1^2 L^3}{6EI}$$

Consider *BC* : (*x* measured from *B*)

$$M_x = -P_1 L - P_2 x$$

$$U_{BC} = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_0^L (-P_1 L - P_2 x)^2 dx$$

$$= \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI}$$

Thus

$$U = U_{AB} + U_{BC}$$

$$= \frac{P_1^2 L^3}{6EI} + \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI}$$

$$= \frac{L^3}{6EI}(4P_1^2 + 3P_1 P_2 + P_2^2)$$

The displacement in the direction 1 due to unit load applied in 2 is

$$\Delta_{12} = \left.\frac{\partial U}{\partial P_1}\right|_{P_1=0, P_2=1}$$

$$= \left.\frac{L^3}{6EI}(8P_1 + 3P_2 + 0)\right|_{P_1=0, P_2=1}$$

$$= \frac{L^3}{2EI}$$

The displacement in the direction 2 due to unit load applied in 1 is

$$\Delta_{21} = \left.\frac{\partial U}{\partial P_2}\right|_{P_1=1, P_2=0}$$

$$= \frac{L^3}{6EI}\left(0 + 3P_1 + 2P_2\right)\Big|_{P_1=1, P_2=0}$$

$$= \frac{L^3}{2EI}$$

Since $\Delta_{12} = \Delta_{21}$, proves the Maxwell-Betti law of reciprocal displacements.

Recap

In this course you have learnt the following

- Maxwell-Betti Law of reciprocal deflection.

- Illustrative examples for proving law of reciprocal deflection.