Lecture – 33

LQG Design; Neighbouring Optimal Control & Sufficiency Condition

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References


Robust Control Design Through Linear Quadratic Gaussian (LQG) Design

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A Practical Control System

Controller → Plant → output

Controller → State Estimation

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Philosophy of LQG Design

- Controller: Linear Quadratic Regulator (LQR)
- State Estimation: Kalman Filter

LQG Design

- LQ Controller (LQR)
- LQ Estimator (Kalman Filter)

LQR Design: Summary

- Performance Index (to minimize):

\[
J = \frac{1}{2} \int_0^\infty \left( X^T Q X + U^T R U \right) dt
\]

where \( Q \geq 0 \) (psdf), \( R > 0 \) (pdf)

- System dynamics: \( \dot{X} = AX + BU \)

- Boundary conditions: \( X(0) = X_0 \) : Specified
**LQR Design: Summary**

- Optimal control

\[ U = -\left( R^{-1} B^T P \right) X = -K X \]

where \( P \) is the solution of

\[ PA + A^T P - PBR^{-1}B^T P + Q = 0 \]

(Algebraic Riccati Equation)

**Kalman Filter Design: Summary**

- **Goal:** To obtain an estimate of the state vector using the state dynamics as well as a ‘sequence of measurements’ as accurate as possible.

- **System Dynamics:** \( \dot{X} = AX + BU + GW \)
  where ‘\( W \)’ is the process noise vector.

- **Measured Output:** \( Y = CX + V \)

- **Assumption:** \( W \) and \( V \) are “white noise”

- **Definitions:**
  \[ Q = E[W W^T], \quad R = E[V V^T] \]
**Kalman Filter Design: Summary**

- Initialize: \( \hat{X}(0) \)
- Solve for Riccati matrix P from the Filter ARE:
  \[
  AP + PA^T - PC^T R^{-1} CP + GQG^T = 0
  \]
- Compute Kalman Gain:
  \[
  K_e = PC^T R^{-1}
  \]
- Propagate the Filter Dynamics:
  \[
  \dot{\hat{X}} = A\hat{X} + BU + K_e (Y - C\hat{X})
  \]

**LQG Design**

- Design a deterministic LQR control \( U = -KX \), assuming perfect knowledge of the states and assuming that the plant is not affected by process and sensor noises.
- Design a Kalman Filter to estimate the states and compute the control using this estimated states \( U = -K\hat{X} \). This design philosophy is called Linear Quadratic Gaussian (LQG) design.
- Justification for the LQG design comes from the “Separation Principle”.

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Separation Theorem in LQG Design

System dynamics:
\[
\dot{X} = AX + BU + GW = AX - BK\dot{X} + GW
= AX - BK(X - \dot{X}) + GW
= (A - BK)X + BK\dot{X} + GW
\]

Error dynamics in Kalman filter:
\[
\dot{\hat{X}} = (A - K_C)\hat{X} + \left(GW - K_C\nu\right)
\]

Combined dynamics:
\[
\begin{bmatrix}
  \dot{X} \\
  \dot{\hat{X}}
\end{bmatrix} =
\begin{bmatrix}
  A - BK & BK \\
  0 & A - K_C
\end{bmatrix}
\begin{bmatrix}
  X \\
  \hat{X}
\end{bmatrix}
+ \begin{bmatrix}
  GW \\
  GW - K_C\nu
\end{bmatrix}
\]

Separation Theorem in LQG Design

Combined expected dynamics:
\[
E\left[\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \end{bmatrix}\right] =
\begin{bmatrix}
  A - BK & BK \\
  0 & A - K_C
\end{bmatrix}
E\left[\begin{bmatrix} X \\ \hat{X} \end{bmatrix}\right] +
E\left[\begin{bmatrix}
  GW \\
  GW - K_C\nu
\end{bmatrix}\right]
\]

\[
\frac{d}{dt}E\left(\begin{bmatrix} X \\ \hat{X} \end{bmatrix}\right) =
\begin{bmatrix}
  A - BK & BK \\
  0 & A - K_C
\end{bmatrix}
E\left(\begin{bmatrix} X \\ \hat{X} \end{bmatrix}\right)
\]

Poles of the combined expected dynamics are dictated by the following characteristic equation:
\[
\begin{bmatrix}
  sI - (A - BK) & -BK \\
  0 & sI - (A - K_C)
\end{bmatrix} =
\begin{bmatrix}
  sI - (A - BK) \\
  0
\end{bmatrix}
\begin{bmatrix}
  0 & sI - (A - K_C)
\end{bmatrix} = 0
\]

Hence, the poles of this system are poles of the controller and the poles of the filter.
Hence, the controller and the filter can be designed separately!
Short Period Control Using LQG Design

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System Dynamics

\( F \) - 16 longitudinal dynamics: \( \dot{X} = AX + B\delta + Gw \)

\[ X = [\alpha \quad q]^T \]

\( \alpha = \) Angle of attack
\( q = \) Pitch rate
\( \delta = \) Elevator deflection
\( w_g = \) Vertical wind gust disturbance

\[
A = \begin{bmatrix}
-1.01887 & 0.90506 \\
0.82225 & -1.07741
\end{bmatrix};
B = \begin{bmatrix}
-0.00215 \\
-0.17555
\end{bmatrix};
G = \begin{bmatrix}
0.00203 \\
-0.00164
\end{bmatrix}
\]
Augmented System with Shaping Filter

The shaping filter dynamic: $\dot{Z} = A_w Z + B_w w$

where, $w$ = white noise

$w_g = C_w Z$

where, $w_g$ = Gust noise

$$A_w = \begin{bmatrix} 0 & 1 \\ -0.0823 & -0.5737 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_w = \begin{bmatrix} 2.1728 & 13.1192 \end{bmatrix}$$

The augmented system dynamics:

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} A & GC_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \delta + \begin{bmatrix} CD_w \\ B_u \end{bmatrix} w$$

The elevator actuator with transfer function:

$$\frac{20.2}{(s + 20.2)}$$

Augmented System with Shaping Filter

$$\dot{X}_{\text{ass}} = A_{\text{ass}} X_{\text{ass}} + B_{\text{ass}} u + G_{\text{ass}} w$$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\vartheta} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1.01887 & 0.90506 & 0.00441 & 0.02663 & -0.00215 \\ 0.82225 & -1.07741 & -0.00365 & -0.02152 & -0.17555 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -0.0823 & -0.5737 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \vartheta \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u + 0 \\ w \end{bmatrix}$$

$u$ = Elevator actuator input

The measured output:

$$Y = \begin{bmatrix} n_i \\ q \end{bmatrix} = CX_{\text{ass}} + V = \begin{bmatrix} 15.87875 & 1.48113 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} X_{\text{ass}} + V$$

$n_i$ = Normal acceleration = $15.87875 \alpha + 1.48113 q$

$V$ = Measurement noise vector
Covariance Matrices and Controller design

The measurement noise covariance: \( R = \begin{bmatrix} 0.64 & 0 \\ 0 & 0.49 \end{bmatrix} \)

The process noise covariance: \( Q = \sigma^2 = 25 \)

The controller design is based on LQR control design.

\[
\begin{align*}
u &= -KX_{\text{aug}} = -R_{\text{amp}}^{-1}B_{\text{amp}}^TP_{\text{amp}}X_{\text{aug}} \\
P_{\text{amp}} &\text{ will find out from the Algebraic Riccati Equation (ARE):} \\
P_{\text{amp}}A_{\text{aug}}^T + A_{\text{aug}}P_{\text{amp}} - P_{\text{amp}}B_{\text{aug}}R_{\text{amp}}^{-1}B_{\text{amp}}^TP_{\text{amp}} = 0
\end{align*}
\]

where the cost function: 
\[
J = \frac{1}{2} \int_0^\infty (X_{\text{aug}}^TQ_{\text{amp}}X_{\text{aug}} + u^TR_{\text{amp}}u)dt
\]

Results

Angle of attach (\( \alpha \)) Vs. Time

Pitch rate (\( \dot{q} \)) Vs. Time
Results

Control Vs. Time

Problem of LQG design & Solution

- Problem
  - Loss of Robustness

- Solution
  - LTR Design
  - Often implemented as LQG/LTR design

- Extension Ideas: $H_2 / H_\infty$ Designs
Optimal Control Problem

- Performance Index (PI): \( J = \varphi(X_f) + \int_{t_0}^{t_f} L(t, X, U) \, dt \)
- Path Constraint: \( \dot{X} = f(t, X, U) \)
- Boundary Conditions: \( X(0) = X_0 \), \( X(t_f) = X_f \) : Specified; \( t_f \) : Fixed, \( \psi(X_f) = 0 \) (q equations)
- Augmented PI:
  \[
  \tilde{J} = [\varphi(X_f) + V^T \psi(X_f)] + \int_{t_0}^{t_f} \left[ L + \dot{\lambda}^T (f - \dot{X}) \right] \, dt
  \]
Necessary Conditions of Optimality

• State Equation
  \[ \dot{X} = \frac{\partial H}{\partial \lambda} = f(t, X, U) \]

• Costate Equation
  \[ \lambda = -\left( \frac{\partial H}{\partial X} \right) \]

• Optimal Control Equation
  \[ \frac{\partial H}{\partial U} = 0 \]

• Boundary Condition
  \[ \lambda_f = \frac{\partial \Phi}{\partial X_f} + \nu^T \frac{\partial \psi}{\partial X_f}, \quad X(t_0) = X_0 : \text{Fixed} \]

Neighbouring Optimal Control: Problem Formulation

Assumption:
We have determined a control solution \( U(t) \), satisfying all necessary conditions.

Let us consider "small perturbations" in the extremal path, produced by small perturbations in the initial state \( \delta X_0 \) and terminal condition \( \delta \psi \).

Questions:
1) Under what conditions \( U(t) \) is \textbf{guaranteed} to be a local optimum?
2) Can we find the neighbouring optimal solution (using the available optimal solution) in an "efficient manner"?
3) Under what condition(s), such a neighbouring solution exists?
**Neighbouring Optimal Control**

Note that the available control solution satisfies all the necessary conditions of optimality; i.e. it makes $\delta J = 0$. Hence, to address our problem, we will have to consider the "second variation", which is given by:

$$
\delta^2 J = \frac{1}{2} \left[ \delta X^T \left( \frac{\partial^2}{\partial X \partial X} + \left( v^T \psi_x \right)_x \right) \delta X \right] + \frac{1}{2} \left[ \delta U^T \frac{\partial^2}{\partial U \partial U} \delta U \right] dt
$$

With respect to the perturbations, the deviation dynamics can be written as:

$$
\delta \dot{X} = \left[ f_x \right] \delta X + \left[ f_u \right] \delta U
$$

Similarly, the deviation in boundary conditions can be written as:

$$
\delta X_0 : \text{Specified, } \left( \psi_x \delta X \right)_0 = \delta \psi : \text{Specified}
$$

**Observations:** The problem appears as a "linear quadratic regulator (LQR) problem with cross-product terms" between the state and control.

**Necessary Conditions:**

1) State Equation: 
$$
\delta \dot{X} = \left[ f_x \right] \delta X + \left[ f_u \right] \delta U
$$

2) Costate Equation: 
$$
\delta \lambda = -H_{XX} \delta X + H_{XU} \delta U - H_{Xx} \delta \lambda
$$

3) Optimal Control Equation: 
$$
0 = H_{Uu} \delta U + H_{UX} \delta X + H_{Ux} \delta \lambda
$$

4) Boundary Conditions: 
$$
\delta X_0 : \text{Specified, } \left[ \psi_x \delta X \right]_0 = \delta \psi : \text{Specified}
$$
Neighbouring Optimal Control

Optimal Control Equation: \[ \delta U = -H^{-1}_{uu} \left[ H_{ux} \delta X + f_u \delta \lambda \right] \]

State and Costate Equations:
\[
\begin{bmatrix}
\delta \dot{X} \\
\delta \dot{\lambda}
\end{bmatrix} =
\begin{bmatrix}
A(t) & -B(t) \\
-C(t) & -A^T(t)
\end{bmatrix}
\begin{bmatrix}
\delta X \\
\delta \lambda
\end{bmatrix}
\]

where
\[
\begin{align*}
A(t) & \triangleq f_x - f_u H^{-1}_{uu} H_{ux} \\
B(t) & \triangleq f_u H^{-1}_{uu} f_u^T \\
C(t) & \triangleq H_{xx} - H_{ux} H^{-1}_{uu} H_{ux}
\end{align*}
\]

(Note: \(B^T = B\))

Seek the solution of \(\delta \lambda\) and \(\delta \psi\) as
\[
\delta \lambda = P(t) \delta X + R(t) d\nu, \quad P > 0 \quad \text{(pdf matrix)}
\]
\[
\delta \psi = R^T(t) \delta X + Q(t) d\nu
\]

(Note: \(\delta \psi\) and \(d\nu\) are infinitesimal "constant vectors")

Boundary Conditions:
\[
\delta x_i = P(t_i) \delta x_i + R(t_i) d\nu = \left[ \left( \varphi_{xx} + \left( v^T \varphi_x \right) x \right) \delta X + \varphi_{x}^T d\nu \right]_{t_i}
\]
\[
\delta \psi = R^T(t_i) \delta X + Q(t_i) d\nu = \left[ \varphi_x \delta X \right]_{t_i}
\]

This gives
\[
\begin{align*}
P(t_i) &= \left( \varphi_{xx} + \left( v^T \varphi_x \right) x \right)_{t_i} \quad , \quad R(t_i) = \left( \varphi_{x}^T \right)_{t_i} \quad , \quad Q(t_i) = 0
\end{align*}
\]
Neighbouring Optimal Control

Next, differentiating $\delta \lambda$ and $\delta \nu$, we obtain

$$\delta \lambda = \dot{P} \delta X + P \delta \dot{X} + \dot{R} \delta \nu$$

$$0 = \dot{R}^T \delta X + R^T \delta \dot{X} + \dot{Q} \delta \nu \quad \text{(Note: $\delta \nu$ and $d\nu$ are constant vectors)}$$

However,

$$\begin{bmatrix} \delta \dot{X} \\ \delta \lambda \end{bmatrix} = \begin{bmatrix} A(t) & -B(t) \\ -C(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} \delta X \\ \delta \lambda \end{bmatrix}$$

Hence

$$\delta \dot{\lambda} = \dot{P} \delta X + P \delta \dot{X} + \dot{R} \delta \nu$$

$$= \dot{P} \delta X + P \left( A \delta X - B \delta \lambda \right) + \dot{R} \delta \nu$$

$$- C \delta X - A^T \left( P \delta X + R \delta \nu \right) = \dot{P} \delta X + P \left( A \delta X - B \left( P \delta X + R \delta \nu \right) \right) + \dot{R} \delta \nu$$

$$\left( \dot{P} + PA - PBP + A^T P + C \right) \delta X + \left( \dot{R} - PBR + A^T R \right) \delta \nu = 0 \quad \cdots(1)$$

Similarly, $0 = \dot{R}^T \delta X + R^T \delta \dot{X} + \dot{Q} \delta \nu$

$$= \dot{R}^T \delta X + R^T \left( A \delta X - B \left( P \delta X + R \delta \nu \right) \right) + \dot{Q} \delta \nu$$

$$0 = \left[ \dot{R}^T + R^T \left( A - BP \right) \right] \delta X + \left[ \dot{Q} - R^T BR \right] \delta \nu \quad \cdots(2)$$

From equations (1) and (2), we obtain

$$\begin{align*}
\dot{P} + PA - PBP + A^T P + C &= 0 \\
\dot{R} - PBR + A^T R &= 0 \\
\dot{Q} - R^T BR &= 0
\end{align*}$$

Differential equations and boundary conditions are now available for solving $P, Q, R$ matrices.

$$P(0) = \varphi_x, \quad P(0) = \varphi_x'$$

$$Q(0) = 0$$

$$Q(0) = 0$$
Neighbouring Optimal Control

After integrating the equations from $t_f$ to $t$, the $d\nu$ value can be computed as

$$d\nu = [Q(t_0)]^{-1} \left[ \delta \psi - R^T (t_0) \delta X_0 \right]$$

Hence, the existence of $d\nu$ for all $\delta \psi$ depends on the non-singularity of $Q(t_0)$.

If "$Q(t_0)$ is singular", then the optimization problem is said to be "abnormal" and in that case the neighbouring optimal solution does not exist.

However, assuming the problem to be normal (i.e. $Q(t_0)$ to be non-singular),

$$d\lambda = P_0 \delta X_0 + R_n d\nu$$

$$= P_0 \delta X_0 + R_n Q_0^{-1} \left[ \delta \psi - R_0^T \delta X_0 \right]$$

$$= (P_0 - R_n Q_0^{-1} R_0) \delta X_0 + R_n Q_0^{-1} \delta \psi$$

Neighbouring Optimal Control

Note that $d\nu$ was evaluated at $t_0$. In terms of a feedback law, however, $d\nu$ can be evaluated at the current time $t$ as

$$d\nu = [Q(t)]^{-1} \left[ \delta \psi - R^T (t) \delta X \right]$$

Finally, the control expression is

$$\delta U = -H_{u}^{-1} \left[ H_{ux} \delta X + f_u^T \delta \alpha \right]$$

$$= -H_{u}^{-1} \left[ H_{ux} \delta X + f_u^T \left( P \delta X + R d\nu \right) \right]$$

$$= -H_{u}^{-1} \left[ H_{ux} + f_u^T P \right] \delta X - H_{u}^{-1} f_u^T R Q^{-1} \left[ \delta \psi - R^T \delta X \right]$$

$$= -H_{u}^{-1} \left[ H_{ux} + f_u^T P - f_u^T R Q^{-1} R \right] \delta X - \frac{1}{K_i(t)} \frac{\delta \psi}{\delta X}$$

$$= -K_i(t) \delta X - K_i(t) \delta \psi$$ : A linear feedback law
Neighbouring Optimal Control: Simplified Version

In many problems, the constraint $\psi(X) = C$ is not critical, and hence, is not imposed. Assuming that $[\varphi_{XX}]_\psi = S_\psi \geq 0$ (a psdf matrix), the expression for $\delta T$ becomes

$$
\delta^2 T = \frac{1}{2} \begin{bmatrix} \delta X^T \, S_\psi \, \delta X + \frac{1}{2} \int_0^T & \delta X^T \, \delta U \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XX} \\ H_{UX} & H_{UU} \end{bmatrix} \begin{bmatrix} \delta X \\ \delta U \end{bmatrix} dt
$$

This leads to a regular LQR problem with cross-product term...we know its solution!

Furthermore, as far as the neighboring optimal solution is concerned, to simplify numerical computation, it is imposed that $t \rightarrow \infty$ with artificial increase in weights on $\delta X$ and $\delta U$ (to have a somewhat similar effect as a finite-time problem). In that case, the problem boils down to the regular $\infty$ – time LQR problem, which is solved by solving the Algebraic Riccati Equation (ARE) online (SDRE formulation).

Sufficiency Condition for Optimality

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Sufficiency Condition

- In weak sense: when $\delta X$ and $\delta \dot{X}$ are small.

- In strong sense: when $\delta X$ is small.

Here the conditions in “weak sense” only are summarized. (see References for conditions in “strong sense”)

**Theorem-1 (existence for neighbouring optimum path)**

The neighbouring optimum paths exist in a weak sense, if $\forall t \in [t_0, t_f]$, the following conditions are satisfied:

1. $H_{00}(t) > 0$ (a pdf matrix) : Convexity condition
2. $Q(t) < 0$ (a ndf matrix) : Normality condition
3. $[P(t) - R(t)Q^{-1}(t)R^T(t)]$ is finite: Jacobi condition

Note that condition (3) is a substitute for the exact condition, which requires that there is no “conjugate point” on the optimal path.

**Theorem-2 (sufficiency condition for minimization)**

The conditions in Theorem - 1, along with the necessary conditions, form a set of sufficient conditions for a trajectory to be local minimum. 
References


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Thanks for the Attention....!!