Lecture – 21

Pole Placement Control Design

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Pole Placement Control Design

Assumptions:

- The system is completely state controllable.
- The state variables are measurable and are available for feedback.
- Control input is unconstrained.
Pole Placement Control Design

Objective:
The closed loop poles should lie at $\mu_1, \ldots, \mu_n$, which are their ‘desired locations’.

Difference from classical approach:
Not only the “dominant poles”, but “all poles” are forced to lie at specific desired locations.

Necessary and sufficient condition:
The system is completely state controllable.
**Closed Loop System Dynamics**

\[
\dot{X} = AX + BU
\]

The control vector \( U \) is designed in the following state feedback form

\[
U = -KX
\]

This leads to the following closed loop system

\[
\dot{X} = (A - BK)X = A_{CL}X
\]

where \( A_{CL} \triangleq (A - BK) \)
Philosophy of Pole Placement Control Design

The gain matrix $K$ is designed in such a way that

$$\left| sI - (A - BK) \right| = (s - \mu_1)(s - \mu_2) \cdots (s - \mu_n)$$

where $\mu_1, \cdots, \mu_n$ are the desired pole locations.
Pole Placement Design Steps: Method 1 (low order systems, $n \leq 3$)

- Check controllability
- Define $K = [k_1, k_2, k_3]$
- Substitute this gain in the desired characteristic polynomial equation:
  $$|sI - A + BK| = (s - \mu_1) \cdots (s - \mu_n)$$
- Solve for $k_1, k_2, k_3$ by equating the like powers on both sides
### Pole Placement Control Design: Method – 2

\[ \dot{X} = AX + Bu \]

\[ u = -KX, \quad K = [k_1 \ k_2 \cdots k_n] \]

Let the system be in first companion (controllable canonical) form

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & 1 \\
a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & -a_1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
\]
After applying the control, the closed loop system dynamics is given by

\[
\dot{X} = (A - BK)X = A_{CL}X
\]

\[
A_{CL} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \\
\end{bmatrix} - \begin{bmatrix}
k_1 & k_2 & k_3 & \ldots & k_n \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & 1 \\
(-a_n - k_1) & (-a_{n-1} - k_2) & \ldots & \ldots & (-a_1 - k_n) \\
\end{bmatrix}
\]

\[\text{(1)}\]
If $\mu_1, \cdots, \mu_n$ are the desired poles. Then the desired characteristic polynomial is given by,

$$(s - \mu_1) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

This characteristic polynomial, will lead to the closed loop system matrix as

$$A_{CL} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1
\end{bmatrix} \cdots (2)$$

State space form
Comparing Equation (1) and (2), we arrive at:

$$
\begin{align*}
    a_n + k_1 &= \alpha_n \\
    a_{n-1} + k_2 &= \alpha_{n-1} \\
    \vdots \\
    a_1 + k_n &= \alpha_1
\end{align*}
\Rightarrow
\begin{align*}
    k_1 &= (\alpha_n - a_n) \\
    k_2 &= (\alpha_{n-1} - a_{n-1}) \\
    \vdots \\
    k_n &= (\alpha_1 - a_1)
\end{align*}
$$

$$K = (\alpha - a) \quad \text{(Row vector form)}$$
What if the system is not given in the first companion form?

Define a transformation \( X = T\dot{X} \)

\[
\begin{align*}
\dot{X} &= T^{-1}\dot{X} \\
\dot{X} &= T^{-1}(AX + Bu) \\
\dot{X} &= (T^{-1}AT)\dot{X} + (T^{-1}B)u
\end{align*}
\]

Design a \( T \) such that \( T^{-1}AT \) will be in first companion form.

Select \( T = MW \)

where \( M \triangleq \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \) is the controllability matrix
Next, design a controller for the transformed system (using the technique for systems in first companion form).

\[ u = -\hat{K}\hat{X} = -(\hat{K}T^{-1})X = -KX \]

**Note:** Because of its role in control design as well as the use of (Controllability Matrix) in the process, the ‘first companion form’ is also known as ‘Controllable Canonical form’.
Pole Placement Design Steps: Method 2: Bass-Gura Approach

- Check the controllability condition
- Form the characteristic polynomial for $A$
  $$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$
  find $a_i$’s
- Find the Transformation matrix $T$
- Write the desired characteristic polynomial
  $$(s - \mu_1) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$
  and determine the $\alpha_i$’s
- The required state feedback gain matrix is
  $$K = [(\alpha_n - a_n) \ (\alpha_{n-1} - a_{n-1}) \ \cdots \ (\alpha_1 - a_1)] T^{-1}$$
Pole Placement Design Steps: Method 3 (Ackermann’s formula)

Define $\tilde{A} = A - BK$

desired characteristic equation is

$$|sI - (A - BK)| = (s - \mu_1) \cdots (s - \mu_n)$$

$$|sI - \tilde{A}| = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

Caley-Hamilton theorem states that every matrix $A$
satisfies its own characteristic equation

$$\phi(\tilde{A}) = \tilde{A}^n + \alpha_1 \tilde{A}^{n-1} + \alpha_2 \tilde{A}^{n-2} + \cdots + \alpha_{n-1} \tilde{A} + \alpha_n = 0$$

For the case $n = 3$ consider the following identities.

\[
\begin{align*}
I &= I \\
\tilde{A} &= A - BK \\
\tilde{A}^2 &= (A - BK)^2 = A^2 - ABK - BK\tilde{A} \\
\tilde{A}^3 &= (A - BK)^3 = A^3 - A^2BK - ABK\tilde{A} - BK\tilde{A}^2
\end{align*}
\]
Pole Placement Design Steps: Method 3: (Ackermann’s formula)

Multiplying the identities in order by $\alpha_3, \alpha_2, \alpha_1$ respectively and adding we get

$$\alpha_3 I + \alpha_2 \tilde{A} + \alpha_1 \tilde{A}^2 + \tilde{A}^3$$

$$= \alpha_3 I + \alpha_2 (A - BK) + \alpha_1 (A^2 - ABK - BK\tilde{A}) + A^3 - A^2 BK$$

$$- ABK\tilde{A} - BK\tilde{A}^2$$

$$= \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\tilde{A} - A^2 BK$$

$$- ABK\tilde{A} - BK\tilde{A}^2$$

\[ \text{.........(1)} \]

From Caley-Hamilton Theorem for $\tilde{A}$

$$\alpha_3 I + \alpha_2 \tilde{A} + \alpha_1 \tilde{A}^2 + \tilde{A}^3 = \phi(\tilde{A}) = 0$$

And also we have for $A$

$$\alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 = \phi(A) \neq 0$$
Pole Placement Design Steps: Method 3 (Ackermann’s formula)

Substituting $\varphi(\tilde{A})$ and $\varphi(A)$ in equation (1) we get

\[
\varphi(\tilde{A}) = \varphi(A) - \alpha_2 B K - \alpha_1 B K \tilde{A} - B K \tilde{A}^2 - \alpha_1 A B K - A B K \tilde{A} - A^2 B K
\]

Substituting (3) and (4) in equation (1) we get

\[
\varphi(A) = B (\alpha_2 K + \alpha_1 K \tilde{A} + K \tilde{A}^2) + A B (\alpha_1 K + K \tilde{A}) + A^2 B K
\]

Since system is completely controllable inverse of the controllability matrix exists we obtain

\[
[B \ AB \ A^2 B]^{-1} \varphi(A) = \begin{bmatrix} \alpha_2 K + \alpha_1 K \tilde{A} + K \tilde{A}^2 \\ \alpha_1 K + K \tilde{A} \\ K \end{bmatrix}
\]
Pole Placement Design Steps: Method 3 (Ackermann’s formula)

Pre multiplying both sides of the equation (2) with \([0 \ 0 \ 1]\)

\[
[0 \ 0 \ 1][B \ AB \ A^2B]^{-1}\phi(A) = [0 \ 0 \ 1]
\begin{bmatrix}
\alpha_2K + \alpha_1K\bar{A} + K\bar{A}^2 \\
\alpha_1K + K\bar{A} \\
K
\end{bmatrix} = K
\]

- For an arbitrary positive integer \(n\) (number of states), Ackermann’s formula for the state feedback gain matrix \(K\) is given by

\[
K = [0 \ 0 \ 0 \ \cdots \ \cdots \ 1]\begin{bmatrix} B & AB & A^2B & \cdots & \cdots & A^{n-1}B \end{bmatrix}^{-1} \phi(A)
\]

where \(\phi(A) = A^n + \alpha_1A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_nI\)

\(\alpha_i\)'s\ are\ the\ coefficients\ of\ the\ desired\ characteristic\ polynomial
Choice of closed loop poles: Guidelines

- Do not choose the closed loop poles far away from the open loop poles, otherwise it will demand high control effort.

- Do not choose the closed loop poles very negative, otherwise the system will be fast reacting (i.e., it will have a small time constant).
  - In frequency domain it leads to large bandwidth, and hence noise gets amplified.
Choice of closed loop poles: Guidelines

- Use “Butterworth polynomials”

\[
\left( \frac{s}{w_0} \right) = (-1)^{\frac{n+1}{2n}} \left( e^{(j(2k+1)\pi)} \right)^{\frac{n+1}{2n}} \quad k = 0, 1, 2 - - -
\]

- 

\( w_0 \) = a constant (like "natural frequency")

- 

\( n \) = system order (number of closed loop poles)

- 

choose only stable poles.

Example: 1

- 

let \( n = 1 \) only one pole

- 

use \( k = 1 \)

- 

\( s = \omega_0 (\cos \pi + j \sin \pi) = -\omega_0 \)
Choice of closed loop poles:
Guidelines

Example 2:
Let $n = 2$ we know $(\cos \theta + j\sin \theta)^n = (\cos m\theta + j\sin m\theta)$

\[
\frac{n+1}{2n} = \frac{3}{4}
\]

$s = \omega_0[\cos((2k+1)3\pi/4) + j\sin((2k+1)(3\pi/4))]$

$k = 0 \Rightarrow s_1 = \omega_0[\cos(3\pi/4) + j\sin((3\pi/4))]$

stabilizing: accept.

$k = 1 \Rightarrow s_2 = \omega_0[\cos(9\pi/4) + j\sin((9\pi/4))]$

Destabilizing: reject.

$k = 2 \Rightarrow s_3 = \omega_0[\cos(15\pi/4) + j\sin((15\pi/4))]$

Destabilizing: reject.

$k = 3 \Rightarrow s_4 = \omega_0[\cos(21\pi/4) + j\sin((21\pi/4))]$

stabilizing: accept.
Example: Inverted Pendulum

\[
\begin{align*}
\dot{x}_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} \frac{M + m}{Ml} g & 0 & 0 & 0 \end{bmatrix} u \\
\dot{x}_2 &= \begin{bmatrix} \frac{M + m}{Ml} g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\
\dot{x}_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} -\frac{1}{Ml} \end{bmatrix} u \\
\dot{x}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} \frac{1}{M} \end{bmatrix} u \\
y_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_1 \\
y_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x_1 \\
\begin{bmatrix} A & \begin{bmatrix} 0 \\ 20.601 \\ 0 \\ -0.4905 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} &; \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} \\
\begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{align*}
\]
Step 1: Check controllability

\[
M = [B \mid AB \mid A^2B \mid A^3B] = \begin{bmatrix}
0 & -1 & 0 & -20.601 \\
-1 & 0 & -20.601 & 0 \\
0 & 0.5 & 0 & 0.4905 \\
0.5 & 0 & 0.4905 & 0
\end{bmatrix}
\]

\[|M| \neq 0\]

Hence, the system is controllable.
Step 2: Form the characteristic equation and get $a_i$'s

$$|sI - A| = \begin{bmatrix} s & -1 & 0 & 0 \\ -20.601 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 0.4905 & 0 & 0 & s \end{bmatrix}$$

$$= s^4 - 20.601s^2$$

$$= s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$$

$a_1 = 0, \quad a_2 = -20.601, \quad a_3 = 0, \quad a_4 = 0$
Step 3: Find Transformation $T = MW$ and its inverse

$$W = \begin{bmatrix} a_3 & a_2 & a_1 & 1 \\ a_2 & a_1 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -20.601 & 0 & 1 \\ -20.601 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$T = MW = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -9.81 & 0 & 0.5 & 0 \\ 0 & -9.81 & 0 & 0.5 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} & 0 \\ 0 & -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$
Step 4: Find $\alpha_i$’s from desired poles $\mu_1, \mu_2, \mu_3, \mu_4$

\[
\mu_1 = -2 + j2\sqrt{3}, \quad \mu_2 = -2 - j2\sqrt{3}, \quad \mu_3 = -10, \quad \mu_4 = -10
\]

\[
(s - \mu_1)(s - \mu_2)(s - \mu_3)(s - \mu_4) = (s + 2 - j2\sqrt{3})(s + 2 + j2\sqrt{3})(s + 10)(s + 10)
\]
\[
= (s^2 + 4s + 16)(s^2 + 20s + 100)
\]
\[
= s^4 + 24s^3 + 196s^2 + 720s + 1600
\]
\[
= s^4 + \alpha_1s^3 + \alpha_2s^2 + \alpha_3s + \alpha_4 = 0
\]

\[
\alpha_1 = 24, \quad \alpha_2 = 196, \quad \alpha_3 = 720, \quad \alpha_4 = 1600
\]
Step 5: Find State Feedback matrix $K$ and input $u$

$$
K = [\alpha_4 - a_4 \mid \alpha_3 - a_3 \mid \alpha_2 - a_2 \mid \alpha_1 - a_1] T^{-1}
= [1600 - 0 \mid 720 - 0 \mid 196 + 20.601 \mid 24 - 0] T^{-1}
= \begin{bmatrix}
-0.5 & 0 & - \frac{1}{9.81} & 0 \\
0 & -0.5 & 0 & - \frac{1}{9.81} \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
$$

$u = -Kx = 298.1504x_1 + 60.6972x_2 + 163.0989x_3 + 73.3945x_4$
Time Simulation: Inverted Pendulum

- $x_1$ (Theta) versus $t$
- $x_2$ (Theta dot) versus $t$
- $x_3$ (Displacement of Cart) versus $t$
- $x_4$ (Velocity of Cart) versus $t$
Choice of closed loop poles: Guidelines

- Do not choose the closed loop poles far away from the open loop poles. Otherwise, it will demand high control effort.

- Do not choose the closed loop poles very negative. Otherwise, the system will be fast reacting (i.e. it will have a small time constant).

  In frequency domain it leads to a large bandwidth, which in turn leads to amplification of noise!
Multiple input systems

- The gain matrix is not unique even for fixed closed loop poles.
- Involved (but tractable) mathematics

\[
\dot{X} = AX + BX \\
U = -KX
\]

\[
K = \begin{bmatrix}
k_{11} & k_{12} & \cdots & k_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
k_{m1} & k_{m2} & \cdots & k_{mn}
\end{bmatrix}
\]
Multiple input systems: Some tricks and ideas

- Eliminate the need for measuring some $x_j$ by appropriately choosing the closed loop poles.

  \[ u = \mu_1 x_1 + (\mu_2 - \beta) x_2 \]

  select $\mu_2 = \beta$ provided $\beta < 0$

- Relate the gains to proper physical quantities

  \[
  \begin{bmatrix}
  u_x \\
  u_y \\
  \end{bmatrix} =
  \begin{bmatrix}
  g_{11} & 0 & g_{13} & 0 \\
  0 & g_{22} & 0 & g_{24} \\
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y \\
  x \\
  y \\
  \end{bmatrix}
  \]

- Shape eigenvectors: “Eigen structure assignment control”

- Introduce the idea of optimality: “optimal control”
Summary of Pole Placement Technique

Controllable, Linear, Time-invariant System

Multiple Control Variables

Number of Inputs

Single Control Variable

Define a Linear Combination of Control Variables as a New Control Variable

Verify Controllability

Transform the System to Controllable canonical form

State variable feedback Gain Matrix

Closed loop characteristic polynomial

Compare the coefficient of like powers of $s$

Desired pole location

Desired characteristic polynomial

Solve a system of Linear Equations for the gain matrix elements

Control Allocation:
References


Thanks for the Attention...!