Lecture – 33

Stability Analysis of Nonlinear Systems Using Lyapunov Theory – I

Dr. Radhakant Padhi
Asst. Professor
Dept. of Aerospace Engineering
Indian Institute of Science - Bangalore
Outline

- Motivation
- Definitions
- Lyapunov Stability Theorems
- Analysis of LTI System Stability
- Instability Theorem
- Examples
References


Techniques of Nonlinear Control Systems Analysis and Design

- Phase plane analysis
- Differential geometry (Feedback linearization)
- Lyapunov theory
- Intelligent techniques: Neural networks, Fuzzy logic, Genetic algorithm etc.
- Describing functions
- Optimization theory (variational optimization, dynamic programming etc.)
Motivation

- Eigenvalue analysis concept does not hold good for nonlinear systems.
- Nonlinear systems can have multiple equilibrium points and limit cycles.
- Stability behaviour of nonlinear systems need not be always global (unlike linear systems).
- Need of a systematic approach that can be exploited for control design as well.
Definitions

System Dynamics

\[ \dot{X} = f(X) \quad f : D \to \mathbb{R}^n \text{ (a locally Lipschitz map)} \]

\[ D : \text{an open and connected subset of } \mathbb{R}^n \]

Equilibrium Point \( (X_e) \)

\[ \dot{X}_e = f(X_e) = 0 \]
Definitions

**Open Set**  
A set $A \subset \mathbb{R}^n$ is open if  
for every $p \in A$, $\exists B_r(p) \subset A$

**Connected Set**
- A connected set is a set which cannot be represented as the union of two or more disjoint nonempty open subsets.
- Intuitively, a set with only one piece.

Space $A$ is connected, $B$ is not.
Definitions

Stable Equilibrium

\( X_e \) is stable, provided for each \( \varepsilon > 0 \), \( \exists \delta(\varepsilon) > 0 \):
\[
\|X(0) - X_e\| < \delta(\varepsilon) \implies \|X(t) - X_e\| < \varepsilon \quad \forall t \geq t_0
\]

Unstable Equilibrium

If the above condition is not satisfied, then the equilibrium point is said to be unstable.
Definitions

Convergent Equilibrium

If \( \exists \delta : \| X(0) - X_e \| < \delta \) \( \Rightarrow \lim_{t \to \infty} X(t) = X_e \)

Asymptotically Stable

If an equilibrium point is both stable and convergent, then it is said to be asymptotically stable.
Definitions

Exponentially Stable
\[ \exists \alpha, \lambda > 0: \quad \|X(t) - X_e\| \leq \alpha \|X(0) - X_e\| e^{-\lambda t} \quad \forall t > 0 \]

whenever
\[ \|X(0) - X_e\| < \delta \]

Convention

The equilibrium point \( X_e = 0 \)

(without loss of generality)
Definitions

A function $V : D \rightarrow \mathbb{R}$ is said to be positive semi definite in $D$ if it satisfies the following conditions:

(i) $0 \in D$ and $V(0) = 0$

(ii) $V(X) \geq 0$, $\forall X \in D$

$V : D \rightarrow \mathbb{R}$ is said to be positive definite in $D$ if condition (ii) is replaced by $V(X) > 0$ in $D - \{0\}$

$V : D \rightarrow \mathbb{R}$ is said to be negative definite (semi definite) in $D$ if $-V(X)$ is positive definite.
Lyapunov Stability Theorems

Theorem – 1 (Stability)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \rightarrow \mathbb{R}^n$. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$

(ii) $V(X) > 0$, in $D - \{0\}$

(iii) $\dot{V}(X) \leq 0$, in $D - \{0\}$

Then $X = 0$ is "stable".
Lyapunov Stability Theorems

Theorem – 2 (Asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \ f : D \rightarrow \mathbb{R}^n$.

Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$

(ii) $V(X) > 0$, in $D \setminus \{0\}$

(iii) $\dot{V}(X) < 0$, in $D \setminus \{0\}$

Then $X = 0$ is "asymptotically stable".
Lyapunov Stability Theorems

Theorem – 3 (Globally asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \rightarrow \mathbb{R}^n$. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$

(ii) $V(X) > 0$, $in D - \{0\}$

(iii) $V(X)$ is "radially unbounded"

(iv) $\dot{V}(X) < 0$, $in D - \{0\}$

Then $X = 0$ is "globally asymptotically stable".
Lyapunov Stability Theorems

Theorem – 3 (Exponentially stable)

Suppose all conditions for asymptotic stability are satisfied. In addition to it, suppose \( \exists \) constants \( k_1, k_2, k_3, p \):

(i) \( k_1 \| X \|^p \leq V(X) \leq k_2 \| X \|^p \)

(ii) \( \dot{V}(X) \leq -k_3 \| X \|^p \)

Then the origin \( X = 0 \) is "exponentially stable".
Moreover, if these conditions hold globally, then the origin \( X = 0 \) is "globally exponentially stable".
Example: Pendulum Without Friction

- System dynamics
  \[
  \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
  \end{bmatrix}
  =
  \begin{bmatrix}
  x_2 \\
  -\left(\frac{g}{l}\right)\sin x_1
  \end{bmatrix}
  \]

- Lyapunov function
  \[
  V = KE + PE
  = \frac{1}{2} m (\omega l)^2 + mgh
  = \frac{1}{2} ml^2 x_2^2 + mg (1 - \cos x_1)
  \]
Pendulum Without Friction

\[
\dot{V}(X) = (\nabla V)^T f(X) \\
= \begin{bmatrix}
\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2}
\end{bmatrix} \begin{bmatrix} f_1(X) \\ f_2(X) \end{bmatrix}^T \\
= \begin{bmatrix} mgl \sin x_1 \\ ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 \end{bmatrix}^T \\
= mglx_2 \sin x_1 - mglx_2 \sin x_1 = 0
\]

\[
\dot{V}(X) \leq 0 \quad \text{(nsdf)}
\]

Hence, it is a “stable” system.
Pendulum With Friction

Modify the previous example by adding the friction force $k l \dot{\theta}$

$$ma = -mg \sin \theta - kl \dot{\theta}$$

Defining the same state variables as above

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2$$
Pendulum With Friction

\[ \dot{V}(X) = (\nabla V)^T f(X) \]
\[ = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T \]
\[ = \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}^T \]
\[ = -kl^2 x_2^2 \]
\[ \dot{V}(X) \leq 0 \quad (nsdf) \]

Hence, it is also just a “stable” system. (A frustrating result..!)}
Analysis of Linear Time Invariant System

System dynamics: \[ \dot{X} = AX, \quad A \in \mathbb{R}^{n \times n} \]

Lyapunov function: \[ V(X) = X^T PX, \quad P > 0 \text{ (pdf)} \]

Derivative analysis: \[ \dot{V} = \dot{X}^T PX + X^T P\dot{X} \]
\[ = X^T A^T PX + X^T PAX \]
\[ = X^T \left( A^T P + PA \right) X \]
Analysis of Linear Time Invariant System

For stability, we aim for
$$\dot{V} = -X^T Q X \quad (Q > 0)$$

By comparing
$$X^T \left( A^T P + PA \right) X = -X^T Q X$$

For a non-trivial solution
$$PA + A^T P + Q = 0$$

(Lyapunov Equation)
Analysis of Linear Time Invariant System

**Theorem:** The eigenvalues $\lambda_i$ of a matrix $A \in \mathbb{R}^{n \times n}$ satisfy $\text{Re}(\lambda_i) < 0$ if and only if for any given symmetric pdf matrix $Q$, $\exists$ a unique pdf matrix $P$ satisfying the Lyapunov equation.

**Proof:** Please see Marquez book, pp.98-99.

**Note:** $P$ and $Q$ are related to each other by the following relationship:

$$P = \int_{0}^{\infty} e^{A^T t} Q e^{A t} dt$$

However, the above equation is seldom used to compute $P$. Instead $P$ is directly solved from the Lyapunov equation.
Analysis of Linear Time Invariant Systems

- Choose an arbitrary symmetric positive definite matrix $Q \ (Q = I)$
- Solve for the matrix $P$ form the Lyapunov equation and verify whether it is positive definite
- Result: If $P$ is positive definite, then $\dot{V}(X) < 0$ and hence the origin is “asymptotically stable”.
Lyapunov’s Indirect Theorem

Let the linearized system about $X = 0$ be $\Delta \dot{X} = A(\Delta X)$. The theorem says that if all the eigenvalues $\lambda_i$ $(i = 1, \ldots, n)$ of the matrix $A$ satisfy $\text{Re}(\lambda_i) < 0$ (i.e. the linearized system is exponentially stable), then for the nonlinear system the origin is locally exponentially stable.
### Instability theorem

Consider the autonomous dynamical system and assume $X=0$ is an equilibrium point. Let $V : D \to \mathbb{R}$ have the following properties:

(i) $V(0) = 0$

(ii) $\exists X_0 \in \mathbb{R}^n$, arbitrarily close to $X = 0$, such that $V(X_0) > 0$

(iii) $\dot{V} > 0 \quad \forall X \in U$, where the set $U$ is defined as follows

\[
U = \{ X \in D : \|X\| \leq \varepsilon \quad \text{and} \quad V(X) > 0 \}
\]

Under these conditions, $X=0$ is unstable
Summary

- Motivation
- Notions of Stability
- Lyapunov Stability Theorems
- Stability Analysis of LTI Systems
- Instability Theorem
- Examples
References

Thanks for the Attention...!