Module 2: Lecture 1
GOVERNING EQUATIONS OF FLUID MOTION
(Fundamental Aspects)

Descriptions of Fluid Motion

A fluid is composed of different particles for which the properties may change with respect to time and space. This description of fluid motion is somewhat different in comparison to solid body motion where the body can be tracked as it moves. Here, the fluid molecules are not identified as distinct one, rather a reasonably small chunk of fluid molecules are considered as particle for which the continuum assumption is valid. Then, the motion of this chunk is generally described by its velocity. Hence, the fluid velocity at a point is nothing but the velocity of fluid particle present at that point at that particular instant. Many a times, these chunks of molecules move randomly with different velocities. In such cases, the bulk motion of this chunk is often considered as of interest. So, the velocity can be thought of as mass averaged velocity of the system of molecules present in the chunk i.e. the velocity of the centre of mass of the system of molecules. Once, it is clear about what needs to be measured particle or bulk velocity, the entire domain of flow of this quantity (i.e. velocity) is described by two ways. In the first method, the individual fluid particle is studied as a function of time (Lagrangian approach). In the other case, the bulk motion is prescribed as the functions of space and time (Eulerian approach).

In Lagrangian description, any single particle of fluid from the flow is selected and its flow characteristics such as velocity, acceleration, pressure etc. are closely monitored and noted during the entire course of the flow through space. The position of particle at any instant of time becomes a function of its identity and time. In other words, a moving coordinate system is attached to the particle under study. It is equivalent to an observer sitting on a moving train and studying its motion.

The Eulerian approach deals with any fixed point in the space occupied by the fluid. The observations are made on the changes in flow characteristics that take place at that point. So, the coordinate system fixed to the point in space is selected and the attention is focused on the fixed point as the fluid particles pass over it. It is similar to a situation where on observer standing on the ground watches the motion of a moving train.
In order to illustrate these types of motion, let us refer to Fig. 2.1, where the position of a particle is initially located at a point \( s_0(x_0, y_0, z_0, t_0) \) and then changed to another point \( s(x, y, z, t) \) after some time interval. In Langrangian method, all the quantities of interest associated with this particle, are the functions of its identity (initial point) and time. For example, if it is desired to find out the velocity \( (\vec{v}) \) and acceleration \( (\vec{a}) \) of this particle, then the following expressions may be used.

\[
\vec{v} = \frac{\partial \vec{s}}{\partial t}; \quad \vec{a} = \frac{\partial^2 \vec{s}}{\partial t^2}
\] (2.1.1)

In a two-dimensional plane, if the path functions are described by any arbitrary function, \( x = x_0 e^{ct}; y = y_0 e^{-ct} \), then the corresponding velocity and accelerations can be obtained by Eq. (2.1.1).

\[
egin{align*}
u &= \frac{\partial x}{\partial t} = \frac{\partial}{\partial t} (x_0 e^{ct}) = c x_0 e^{ct}; \quad a_x = \frac{\partial^2 x}{\partial t^2} = c^2 x_0 e^{ct} \\
v &= \frac{\partial y}{\partial t} = \frac{\partial}{\partial t} (y_0 e^{-ct}) = -c y_0 e^{-ct}; \quad a_y = \frac{\partial^2 y}{\partial t^2} = c^2 y_0 e^{-ct}
\end{align*}
\] (2.1.2)

Fig. 2.1.1: Description of fluid motion.
When the above results are calculated in *Eulerian* variables, then the velocity and accelerations are expressed with respect to the particular point in the space. So, the Eq. (2.1.2) can be rewritten as given below;

\[
\begin{align*}
    u &= c x \quad ; \quad a_x = c^2 x \\
    v &= -c y \quad ; \quad a_y = c^2 y
\end{align*}
\] (2.1.3)

Out of the two approaches in the study of fluid motion, the *Eulerian* description is the preferred one because of its mathematical simplicity.

**Concepts of System and Control Volume**

The common approaches for handling the properties in a flow field are discussed in the previous section. So, one can either look at individual particles and find out how the properties associated with it change as it moves. The other approach is to concentrate on a finite region of flow and find out how the flow properties change in that region. Based on above descriptions, the concepts of system and control volume are introduced here to tackle the mathematical model of the basic laws in fluid flows.

In fluid mechanics, a *system* is defined as the chunk of fluid particles whose identity does not change during the course of flow. Here, the identity means that the chunk is composed of same fluid particles as it flows. The natural consequence of this definition is that the mass of the system is invariable since it is composed of the same fluid particles. The shaded oval shown in Fig. 2.1.2(a), is considered as the *system* which moves towards left as indicated by an arrow. Although, the particles inside the oval do not change as it moves, but the shape and size of this oval may change during the course of the flow because different particles have different velocities, Moreover, *Lagrangian* approach will be more appropriate for this method of description.
A control volume is a volume or region in space whose identity is not same as fluid can enter and leave through the control surface which encloses this volume (Fig. 2.1.2-b). The shape and size of the control volume may be fixed or can change depending upon the choice of coordinate system used to analyze the flow situation. Here, the Eulerian variables are more suitable for analysis of flow field.

![Fig. 2.1.2: Concept of system and control volume.](image)

**Basic Physical Laws**

In the theory of fluid mechanics, the flow properties of fluid are generally predicted without actually measuring it. If the initial values of certain minimum number of quantities are known, then the values at some other locations can be obtained by using certain fundamental relationships. However, they are very much local in the sense that they cannot be used for different set of conditions. Such relationships are called as empirical laws/formulae and there are certain relationships which are broadly applicable in a general flow field, falling under the category of ‘basic laws’. Pertaining to the theory of fluid mechanics, there are three most relevant basic laws namely;

- Conservation of mass (continuity equation)
- Conservation of momentum (Newton’s second law of motion)
- Conservation of energy (First law of thermodynamics)
- Second law of thermodynamics

All these basic laws involve thermodynamic state relations (equation of state, fluid property relation etc.) for a particular fluid being studied.
**Conservation of mass:** There are two ways to define mass namely, inertial mass and gravitational mass. The first one uses Newton’s second law for definition whereas the second one uses Newton’s law of gravitation. In both the cases, numerical value for mass is the same. If this numerical value does not change when the system undergoes a change, then it is treated as “conservation of mass”. In fluid flow situation, if one chooses a system of fluid particles, then the identity remains the same by definition of system and hence the mass for a system is constant. It does not matter whether any chemical reaction/heating or any other process is taking place within the system. Mathematically, it is represented as the time rate of change of mass of a system is zero.

\[
\left. \frac{dm}{dt} \right|_{\text{system}} = 0 \tag{2.1.4}
\]

**Newton’s second law of motion:** It states that the rate of change of linear momentum of a chunk of fluid mass is equal to the net external force acting on it. For a single particle, Newton’s second law is written as,

\[
\bar{F}_{\text{net}} = m \frac{d\bar{V}}{dt} = m\dddot{a} \tag{2.1.5}
\]

where, \( \bar{F}_{\text{net}} \) is the resultant force on the particle, \( m \) is the mass of the particle, \( \bar{V} \) is the velocity of the particle and \( \dddot{a} \) is the acceleration of the particle. Since a fluid mass consists of number of particles, then the net linear momentum \( \bar{M} \) for \( n \) number of particles is given by,

\[
\bar{M} = m_1\bar{V}_1 + m_2\bar{V}_2 + \ldots + m_n\bar{V}_n = \sum_{i=1}^{n} m_i\bar{V}_i \tag{2.1.6}
\]
This equation can be differentiated with respect to \( t \) to obtain the net force acting on the fluid mass.

\[
\frac{dM}{dt} = \sum_{i=1}^{n} m_i \frac{dV_i}{dt} = \sum_{i=1}^{n} m_i a_i = \sum_{i=1}^{n} F_i^e = \sum_{i=1}^{n} F_i^{ext}
\]  

(2.1.7)

Here, \( F_i^{ext} \) is the net external force acting on \( i^{th} \) particle.

The other way to represent the same law is in terms of angular momentum \( \vec{H} \) and for a fluid mass, it is expressed as,

\[
\vec{H} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{V}_i
\]

(2.1.8)

where, \( \vec{r} \) is the position vector of the particle with respect to certain frame of reference.

The rate of change of angular momentum is given by,

\[
\frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \left( \frac{d\vec{r}_i}{dt} \times m_i \vec{V}_i + \vec{r}_i \times m_i \frac{d\vec{V}_i}{dt} \right) = \sum_{i=1}^{n} \left( \vec{V}_i \times m_i \vec{V}_i + \vec{r}_i \times m_i \frac{d\vec{V}_i}{dt} \right) = \sum_{i=1}^{n} \vec{r}_i \times F_{net} = H_{net}
\]

(2.1.9)

Here, \( H_{net} \) is the net external momentum acting on the fluid mass. In other words, the rate of change of angular momentum of a system of fluid particle is equal to the net external moment on the system.

**First law of thermodynamics**: This law is nothing but the energy conservation law which states that energy can neither be created nor be destroyed but, can be changed from one form to another. If \( \delta Q \) is the heat exchange with the system, \( \delta W \) is the work done by the system and \( dE \) is the change in energy of the system, then they are related by the following expressions for a closed system.

\[
\delta Q = dE + \delta W
\]

(2.1.10)
Second law of thermodynamics: This law introduces a new property i.e. entropy \((S)\) and the change in the entropy \((dS)\) is related to the heat transfer \((\delta Q)\) and the absolute temperature \((T)\).

\[
dS \geq \frac{\delta Q}{T}
\]  
(2.1.11)

This inequality accounts for flow analysis involving losses due to friction, viscous dissipation and any other means of non recoverable losses. Minimizing the loss in available energy in any flow situation is of obvious fact of engineering importance.
Module 2 : Lecture 2  
GOVERNING EQUATIONS OF FLUID MOTION  
(Integral Form-Part I)

Reynolds Transport Theorem (RTT)

The basic physical laws can be applied to flow field to relate various flow properties. The flow domains are generally specified through *Eulerian or Lagrangian* approach. Moreover, the flow variables are generally specified as functions of space and time (*Eulerian description*), while the basic laws are applicable to a closed system of particles. The Reynolds Transport Theorem (RTT) relates the information of control volume to the system of particles.

Consider a control volume (CV) at certain time $t$ which is coinciding with the closed mass system (CMS) as shown in Fig. 2.2.1. The CV is bounded by a control surface (CS) $a-c-b-d$. After a certain time interval $\Delta t$, the CMS moves to a new position shown as $a-c1-b-d1$. During this time interval, the outside fluid enters the control volume through the surface $a-c-b$ and leaves through the surface $b-d-a$. There are three different regions I, II and III in the Fig. 2.2.1. If $B$ is any fluid property and
\( \beta \) is its corresponding intensive property, then, the net change in the property during the time \( \Delta t \) is given by,

\[
\frac{B_i + \Delta t - B_i}{\Delta t} = \left( B_{II,t} + \Delta t + B_{III,t} + \Delta t \right) - \left( B_{I,t} + B_{II,t} \right)
\]

\[
\frac{\Delta t}{\Delta t} + \left( B_{I,t} + \Delta t + B_{II,t} + \Delta t \right) - \left( B_{I,t} + B_{II,t} \right) + \left( B_{III,t} + \Delta t + B_{I,t} + \Delta t \right) \tag{2.2.1}
\]

In Eq. (2.1.10), divide both sides by \( \Delta t \) and take the limits \( \Delta t \to 0 \).

\[
\lim_{\Delta t \to 0} \left( \frac{B + \Delta t - B}{\Delta t} \right) = \lim_{\Delta t \to 0} \left( \frac{B_{CV,t} + \Delta t - B_{CV,t}}{\Delta t} \right) + \lim_{\Delta t \to 0} \left( \frac{B_{III,t} + \Delta t - B_{I,t}}{\Delta t} \right)
\]

\[
\Rightarrow \frac{dB}{dt}_{CMS} = \frac{dB}{dt}_{CV} + \text{Net rate of } B \text{ efflux out of the control surface}
\]

\[
\Rightarrow \frac{d}{dt} \left( \int_{CMS} B dV \right) = \frac{d}{dt} \left( \int_{CV} \beta \rho dV \right) + \int_{CS} \beta \rho \left( \vec{V} \cdot \hat{n} \right) dA
\]

where, \( \vec{V} \) is the velocity of fluid with respect to \( CV \), \( \rho \) is the density and \( dV \) is the differential volume and \( dS \) is the differential area vector. Thus, in words, RTT can be stated as, net rate of change of the total property of the control mass system is equal to the sum of the net rate of change of the total property of the coinciding control volume and net rate of total property efflux out of the control surface.

**Corollary of Reynolds Transport Theorem**

The relation between the system rates of change, control volume surface and volume integrals can be established through the Reynolds Transport Theorem (RTT). There are different ways by which RTT is specified. Let us explore them here.

1. The generalized expression of RTT for a fixed control volume with an arbitrary flow pattern is given by,

\[
\frac{d}{dt} \left( B_{system} \right) = \frac{d}{dt} \left( \int_{CV} \beta \rho dV \right) + \int_{CS} \beta \rho \left( \vec{V} \cdot \hat{n} \right) dA
\]

or,

\[
\frac{d}{dt} \left( B_{system} \right) = \int_{CV} \frac{d}{dt} \left( \beta \rho \right) dV + \int_{CS} \beta \rho \left( \vec{V} \cdot \hat{n} \right) dA \tag{2.2.3}
\]
Here, $B_{system}$ is the any property of the system, $\beta$ is the corresponding intensive property, $\rho$ is the density of the fluid, $\vec{V}$ is the velocity vector of the fluid and $\vec{n}$ is the unit normal vector outwards to the area $dA$. The left hand side term of Eq. (2.2.3) is the time rate of change of any system property $B$. The first term in the right hand side of Eq. (2.2.1) is the change of same property $B$ within the control volume while the second term is the change of flux of $B$ passing through the control surface.

2. If the control volume moves uniformly at a velocity $s \vec{V}$, then an observer fixed to this control volume will note a relative velocity $r \vec{V} = \vec{V} - s \vec{V}$ of the fluid crossing the surface. It may be noted that both $\vec{V}$ and $\vec{s} \vec{V}$ must have the same coordinate system.

The expression for RTT can be represented by the following equation.

$$\frac{d}{dt}(B_{system}) = \frac{d}{dt}(\int_{CV} \beta \rho d\vec{V}) + \int_{CS} \beta \rho (\vec{V} \cdot \vec{n}) dA$$

or, $\frac{d}{dt}(B_{system}) = \int_{CV} \frac{d}{dt}(\beta \rho) d\vec{V} + \int_{CS} \beta \rho (\vec{V} \cdot \vec{n}) dA$ \hspace{1cm} (2.2.4)

When, $\vec{V} = 0$, the above equation reduces to Eq. (2.2.3).

3. Consider the most general situation when the control volume is moving and deforming as well. It means the volume integral in Eq. (2.2.4) must allow the volume elements to distort with time. So, the time derivative must be applied after the integration. So, the RTT takes the form as given below.

$$\frac{d}{dt}(B_{system}) = \frac{d}{dt}(\int_{CV} \beta \rho d\vec{V}) + \int_{CS} \beta \rho (\vec{V} \cdot \vec{n}) dA$$

4. Many fluid flow problems involve the boundaries of control surface as few inlets and exits (denoted by $i$) so that flow field is approximately one-dimensional. Moreover, the flow properties are nearly uniform over the cross section of inlet of exits. So, Eq. (2.2.5) reduces to,

$$\frac{d}{dt}(B_{system}) = \frac{d}{dt}(\int_{CV} \beta \rho d\vec{V}) + \sum (\beta_i \rho_i V_{ri} A_i)_{out} - \sum (\beta_i \rho_i V_{ri} A_i)_{in}$$

or, $\frac{d}{dt}(B_{system}) = \int_{CV} \frac{d}{dt}(\beta \rho) d\vec{V} + \sum (\beta_i \rho_i V_{ri} A_i)_{out} - \sum (\beta_i \rho_i V_{ri} A_i)_{in}$ \hspace{1cm} (2.2.6)
Conservation of Mass

The mathematical form of mass conservation applied to a system is written as,

\[
\left( \frac{dm}{dt} \right)_{\text{system}} = 0
\]  \hspace{1cm} (2.2.7)

- In order to apply RTT for mass conservation, substitute the system property as mass of the system i.e. \( B = m \) so that \( \beta = \frac{dm}{dm} = 1 \). Then Eq. (2.2.5) can be applied to obtain the integral mass conservation law for a generalized deformable control volume.

\[
\left( \frac{dm}{dt} \right)_{\text{system}} = \frac{d}{dt} \left( \int_V \rho dV \right) + \int_{CS} \rho (\vec{V} \cdot \vec{n}) dA = 0
\]  \hspace{1cm} (2.2.8)

- In the case of fixed control volume, Eq. (2.2.8) reduces to,

\[
\int_{CV} \left( \frac{d \rho}{dt} \right) dV + \int_{CS} \rho (\vec{V} \cdot \vec{n}) dA = 0
\]  \hspace{1cm} (2.2.9)

- If the control volume has only one-dimensional inlets and outlets, then one can write Eq. (2.2.9) as,

\[
\int_{CV} \left( \frac{d \rho}{dt} \right) dV + \sum_{i} \left( \rho_i A_i v_i \right)_{\text{out}} - \sum_{i} \left( \rho_i A_i v_i \right)_{\text{in}} = 0
\]  \hspace{1cm} (2.2.10)

- If the flow within the control volume is steady with one-dimensional inlets and outlets, then \( \frac{d \rho}{dt} = 0 \) and Eq. (2.2.9 & 2.2.10) reduces to,

\[
\int_{CS} \rho (\vec{V} \cdot \vec{n}) dA = 0
\]

or, \( \sum_{i} \left( \rho_i A_i v_i \right)_{\text{out}} - \sum_{i} \left( \rho_i A_i v_i \right)_{\text{in}} = 0 \)  \hspace{1cm} (2.2.11)

or, \( \sum \left( m_i \right)_{\text{out}} = \sum \left( m_i \right)_{\text{in}} \)

Eq. (2.2.11) states that the mass flows entering and leaving the control volume for a steady flow balance exactly and called as \textit{continuity equation}. 
- If inlet and outlet are not one-dimensional, one has to compute the mass flow rate by integration over the section.

\[ \dot{m}_{cs} = \int_{cs} \rho (\vec{V} \cdot \hat{n}) dA \]  \hspace{1cm} (2.2.12)

- Again considering a fixed control volume, further simplification is possible if the fluid is treated as incompressible i.e. density variation that are negligible during the course of its motion. In fact, it is quite true for liquids in general practice while for gases, the condition is restricted up to gas velocity less than 30% of the speed of sound. It leads to the simplification of Eq. (2.2.9) where \( \frac{d\rho}{dt} = 0 \) and the density term can come out of the surface integral.

\[ \int_{CS} \rho (\vec{V} \cdot \hat{n}) dA = 0 \]

or, \[ \int_{cs} (\vec{V} \cdot \hat{n}) dA = 0 \]  \hspace{1cm} (2.2.13)

- If the inlets and outlets are approximated as one-dimensional, then Eq. (2.2.13) becomes,

\[ \sum (A V_i)_{out} = \sum (A V_i)_{in} \]

\[ \sum \dot{Q}_{out} = \sum \dot{Q}_{in} \]  \hspace{1cm} (2.2.14)

where, \( \dot{Q}_i = AV_i \) is the volume flow passing through the given cross section. Again, if the cross-sectional area is not one-dimensional, the volume flow rate can be obtained as,

\[ Q_{cs} = \int_{cs} (\vec{V} \cdot \hat{n}) dA \]  \hspace{1cm} (2.2.15)

In this way, the average velocity \( V_{av} \) can be defined such that, when multiplied by the section area, the volume flow rate can be obtained.

\[ V_{av} = \frac{Q}{A} = \frac{1}{A} \int (\vec{V} \cdot \hat{n}) dA \]  \hspace{1cm} (2.2.16)
This is also called as the volume-average velocity. If the density varies across any section, the average density in the same manner.

$$\rho_{av} = \frac{1}{A} \int \rho \, dA$$

(2.2.17)

Since, the mass flow is rate the product of density and velocity, and the average product $$(\rho V)_{av}$$ will take the product of the averages of $$\rho$$ and $$V$$.

$$(\rho V)_{av} = \frac{1}{A} \int (\vec{V} \cdot \vec{n}) \, dA = \rho_{av} V_{av}$$

(2.2.18)
Module 2 : Lecture 3
GOVERNING EQUATIONS OF FLUID MOTION
(Integral Form-Part II)

Linear Momentum Equation

The control-volume mass relation (conservation of mass) involves only velocity and density. The vector directions for velocity only show the flow entering or leaving the control volume. However, many specific flow problems involve the calculations forces/moments and energy associated with the flow. At any case, mass conservation is always satisfied and constantly checked.

The linear momentum equation is mainly governed by Newton’s second law of motion for a system; it states that “the time rate of change of the linear momentum of the system is equal to the sum of external forces acting on the system”. Here, the attention is focused to the arbitrary property i.e. linear momentum which is defined by

\[ B = m\vec{V} \quad \Rightarrow \quad \beta = \frac{dB}{dm} = \vec{V} . \]

Applying RTT to the linear momentum for a deformable control volume,

\[
\frac{d}{dt}(m\vec{V})_{\text{system}} = \sum F = \frac{d}{dt}\left( \int \vec{V} \rho dH \right) + \int \vec{V} \rho (\vec{V}_r \cdot \vec{n}) dA \quad (2.3.1)
\]

In this equation, the fluid velocity vector \( \vec{V} \) is measured with respect to inertial coordinate system and the vector sum of all the forces \( \sum F \) acting on the control volume includes the surface forces acting on all fluids and the body forces acting on the masses within the control volume. Since, Eq. (2.3.1) is a vector relation, the equation has three components and the scalar forms are represented below;

\[
\sum F_x = \frac{d}{dt}\left( \int_{cv} u \rho dH \right) + \int_{cs} u \rho (\vec{V}_r \cdot \vec{n}) dA
\]

\[
\sum F_y = \frac{d}{dt}\left( \int_{cv} v \rho dH \right) + \int_{cs} v \rho (\vec{V}_r \cdot \vec{n}) dA \quad (2.3.2)
\]

\[
\sum F_z = \frac{d}{dt}\left( \int_{cv} w \rho dH \right) + \int_{cs} w \rho (\vec{V}_r \cdot \vec{n}) dA
\]
Here, \( u, v \) and \( w \) are the velocity components in the \( x, y \) and \( z \) directions, respectively. For a fixed control volume, \( \vec{V}_e = \vec{V} \) so that Eq. (2.3.1) reduces to,

\[
\sum \vec{F} = \frac{d}{dt} \left( \int_{c_f} \vec{V} \rho d\mathcal{V} \right) + \int_{c_s} \vec{V} \rho \left( \vec{V} \cdot \vec{n} \right) dA
\]

(2.3.3)

- Similar to “mass flux”, the second term in Eq. (2.3.3) can be represented as momentum flux given by the following equation,

\[
\vec{M}_{cs} = \int_{sec} \vec{V} \rho \left( \vec{V} \cdot \vec{n} \right) dA
\]

(2.3.4)

- If the cross-section is one-dimensional, then \( \vec{V} \) and \( \rho \) are uniform over the area and the result for Eq. (2.3.4) becomes,

\[
\vec{M}_{sec,i} = \vec{V}_i \left( \rho \nu \right) A = \bar{m}_i \vec{V}_i
\]

(2.3.5)

Thus, the Eq. (2.3.3) can be simplified for one-dimensional inlets and outlets as follows;

\[
\sum \vec{F} = \frac{d}{dt} \left( \int_{c_f} \vec{V} \rho d\mathcal{V} \right) + \left[ \sum \bar{m}_i \vec{V}_i \right]_{out} - \left[ \sum \bar{m}_i \vec{V}_i \right]_{in}
\]

(2.3.6)

- In terms of application point of view, the momentum equation can be stated as the vector force on a fixed control volume equals the rate of change of vector momentum within the control volume (first term in RHS of Eq. 2.3.6) plus the vector sum of outlet and momentum fluxes (second term in RHS of Eq. 2.3.6). Generally, the surface forces on a control volume (first term in LHS of Eq. 2.3.6) are due to the pressure and viscous stresses of the surrounding fluid. The pressure forces act normal to the surface and inward while the viscous shear stresses are tangential to the surface.
Angular Momentum Equation

The angular-momentum relation can be obtained for control volume by replacing the variable \( B \) as the angular momentum vector \( \vec{H} \). Since, the fluid particles are non-rigid and have variable velocities, one must calculate the angular momentum by integration of the elemental mass \( dm \). It is in contrast to solids where the angular momentum is obtained through the concept of moment of inertia. At, any fixed point ‘O’, the instantaneous angular momentum and its corresponding intensive properties are given by,

\[
\vec{H}_o = \int_{\text{system}} (\vec{r} \times \vec{V}) \, dm; \quad \vec{p} = \frac{d\vec{H}_o}{dm} = (\vec{r} \times \vec{V})
\]  

(2.3.7)

Here, \( \vec{r} \) is the position vector from the point ‘O’ to the elemental mass \( dm \) and \( \vec{V} \) is the velocity vector of that element. Considering RTT for angular momentum, one can obtain the general relation for a deformable control volume.

\[
\left( \frac{d\vec{H}_o}{dt} \right)_{\text{system}} = \frac{d}{dt} \left[ \int_{\text{cv}} (\vec{r} \times \vec{V}) \rho \, dV \right] + \int_{\text{cs}} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \hat{n}) \, dA
\]  

(2.3.8)

By, angular momentum theorem, the rate of change of angular momentum must be equal to sum of all the moments of all the applied forces about a point ‘O’ for the control volume.

\[
\frac{d\vec{H}_o}{dt} = \sum \vec{M}_o = \sum (\vec{r} \times \vec{F})_o
\]  

(2.3.9)

For a non-deformable control volume Eqs (2.3.8 & 2.3.9) can be combined to obtain the following relation.

\[
\sum \vec{M}_o = \frac{d}{dt} \left[ \int_{\text{cv}} (\vec{r} \times \vec{V}) \rho \, dV \right] + \int_{\text{cs}} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \hat{n}) \, dA
\]  

(2.3.10)

If there are one-dimensional inlets and exits, Eq. (2.3.10) is modified as,

\[
\sum \vec{M}_o = \frac{d}{dt} \left[ \int_{\text{cv}} (\vec{r} \times \vec{V}) \rho \, dV \right] + \left[ \sum \left[ (\vec{r} \times \vec{V}) \dot{m} \right]_{\text{out}} - \sum \left[ (\vec{r} \times \vec{V}) \dot{m} \right]_{\text{in}} \right]
\]  

(2.3.11)
Energy Equation

The first law of thermodynamics for a system states that the rate of increase of the total stored energy of the system is equal to net rate of energy additions by the heat transfer into the system plus net rate of energy addition by work transfer into the system. The mathematical statement for energy equation is given by,

\[
\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt}
\]

or,

\[
\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt}
\]  

(2.3.12)

Now, RTT can be applied to the variable energy \( E \) and the corresponding intensive property becomes \( \beta = \frac{dE}{dm} = \epsilon \). So, for a fixed control volume, energy equation is written as,

\[
\frac{dQ}{dt} - \frac{dW}{dt} = \epsilon = \frac{d}{dt} \left( \int_{CV} e \rho dV \right) + \int_{CS} e \rho \left( \nabla \cdot \vec{n} \right) dA
\]

(2.3.13)

Here, \( Q \) is the energy transfer by heat and \( W \) is the energy transfer by work. They are considered as positive when heat is added to the system or work is done by the system. The system energy (per unit mass) mainly consists of different forms such as internal energy, kinetic energy and potential energy.

\[
e = e_{\text{internal}} + e_{\text{kinetic}} + e_{\text{potential}} = \dot{u} + \frac{1}{2} \dot{v}^2 + gz
\]

(2.3.14)

The energy transfer by heat \( (dQ/dt) \) involves the mode of transfer i.e. conduction/convection/radiation. The time derivatives of work transfer can be represented as,

\[
\dot{W} = \dot{W}_z + \dot{W}_v + \dot{W}_p = \dot{W}_z - \int_{CS} (\tau \cdot \vec{V}) dA + \int_{CS} p (\vec{V} \cdot \vec{n}) dA
\]

(2.3.15)
The shear work due to viscous stresses \( \overline{W}_v \) and work done due to pressure forces \( \overline{W}_p \) occur at the control surface while the shaft work \( \overline{W}_s \) is deliberately obtained by the system. Using Eq. (2.3.15) in (2.3.13), one can obtain the control volume energy equation.

\[
\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt} \left( \int_{CV} e \rho dV \right) + \int_{CS} \left( \frac{e + \frac{p}{\rho}}{\rho} \right) \rho \left( \mathbf{V} \cdot \mathbf{n} \right) dA
\]  

(2.3.16)

Here, the pressure work term is combined with the energy flux term because both involve surface integral. Introducing the thermodynamic property enthalpy \( h = \hat{u} + \frac{p}{\rho} \) that occurs in the fixed control volume, Eq. (2.3.16) becomes,

\[
\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt} \left[ \int_{CV} \left( \hat{u} + \frac{1}{2} V^2 + gz \right) \rho dV \right] + \int_{CS} \left( \hat{h} + \frac{1}{2} V^2 + gz \right) \rho \left( \mathbf{V} \cdot \mathbf{n} \right) dA
\]

(2.3.17)

- If the control volume has number of one-dimensional inlets and outlets, then the surface integral reduces to summation of inlet and outlet fluxes i.e.

\[
\int_{CS} \left( \hat{h} + \frac{1}{2} V^2 + gz \right) \rho \left( \mathbf{V} \cdot \mathbf{n} \right) dA = \sum_{\text{in}} \hat{m} \left( \hat{h} + \frac{1}{2} V^2 + gz \right) - \sum_{\text{out}} \hat{m} \left( \hat{h} + \frac{1}{2} V^2 + gz \right)
\]

(2.3.18)

- If the flow is one-dimensional, steady throughout and only one fluid is involved, then the shaft work is zero. Neglecting viscous work, Eq. (2.3.17) reduces to,

\[
\dot{m} \left[ \hat{h}_{\text{out}} - \hat{h}_{\text{in}} \right] + \left( \frac{V_{\text{out}}^2 - V_{\text{in}}^2}{2} \right) + g \left( z_{\text{out}} - z_{\text{in}} \right) = \dot{Q}
\]

or,

\[
\dot{m} \left( \hat{u}_{\text{out}} - \hat{u}_{\text{in}} \right) + \left( \frac{(p/\rho)_{\text{out}} - (p/\rho)_{\text{in}}}{2} \right) + g \left( z_{\text{out}} - z_{\text{in}} \right) = \dot{Q}
\]

(2.3.19)
When the same equation is applied to an infinitesimally thin control volume, then Eq. (2.3.19) reduces to,

\[ m \left[ \frac{dV}{d} + d\left( \frac{P}{\rho} \right) + d\left( \frac{V^2}{2} \right) + g(dz) \right] = \delta Q \]

or,  \[ d\hat{u} + d\left( \frac{P}{\rho} \right) + d\left( \frac{V^2}{2} \right) + g(dz) = \frac{\delta Q}{m} = \delta q \]

or,  \[ d\hat{u} + p\left( \frac{1}{\rho} \right) dp + d\left( \frac{V^2}{2} \right) + g(dz) = \frac{\delta Q}{m} = \delta q \]

Let us discuss the second law of thermodynamics that introduces the concept of thermodynamic property entropy \( S \) of a system. For all pure substances, the \( T - ds \) relation is very common and holds good for common engineering working fluids such as air and water.

\[ Tds = d\hat{u} + p d\left( \frac{1}{\rho} \right) \]

or, \( d\hat{u} = Tds - p d\left( \frac{1}{\rho} \right) \)

Combining Eqs (2.3.20 & 2.3.21), one can obtain,

\[ \frac{dp}{\rho} + d\left( \frac{V^2}{2} \right) + g(dz) = -(Tds - \delta q) \]

The equality sign in Eq. (2.3.22) holds good for the energy equation based on first law of thermodynamics. However, the more appropriate form of second law of thermodynamics accounts for losses by means an inequality. It states that, the time rate of increase of the entropy of a system must be greater than or at least equal to the sum of ratio of net heat transfer rate into the system to absolute temperature for each particle mass in the system receiving heat from surroundings. This general statement can be written mathematically as,

\[ \frac{dS_{\text{system}}}{dt} \geq \sum \left( \frac{\delta Q}{T} \right)_{\text{system}} \]
At the instant, when the system and control volume are identical, the RHS of Eq. (2.3.23) may be written as,

$$\sum \left( \frac{\delta Q}{T} \right)_{\text{system}} = \sum \left( \frac{\delta Q}{T} \right)_{CV}$$  \hspace{1cm} (2.3.24)

Now, RTT can be applied to the variable entropy $S$ and the corresponding intensive property becomes $\beta = \frac{dS}{dm} = s$. For a fixed non-deforming control volume, the expression of RTT becomes,

$$\frac{dS_{\text{system}}}{dt} = \frac{d}{dt} \left( \int_{CV} s \rho dV \right) + \int_{CS} s \rho (\vec{V} \cdot \vec{n}) dA$$  \hspace{1cm} (2.3.25)

Combination of Eqs (2.3.23, 2.3.24 & 2.3.25) gives,

$$\frac{d}{dt} \left( \int_{CV} s \rho dV \right) + \int_{CS} s \rho (\vec{V} \cdot \vec{n}) dA \geq \sum \left( \frac{\delta Q}{T} \right)_{CV}$$  \hspace{1cm} (2.3.26)

Eq. (2.3.26) can be simplified for steady, one-dimensional flow with single inlet as,

$$m (s_{\text{out}} - s_{\text{in}}) \geq \sum \left( \frac{\delta Q}{T} \right)_{CV}$$  \hspace{1cm} (2.3.27)

Considering the specific entropy $(s)$ and with infinitesimal small control volume at uniform absolute temperature $(T)$, Eq. (2.3.27) is simplified as

$$Tds \geq \delta q$$

or, $$Tds - \delta q \geq 0$$  \hspace{1cm} (2.3.28)

The equality sign holds good for any reversible (frictionless) process while the inequality sign is applicable for irreversible processes involving friction.
Module 2 : Lecture 4
GOVERNING EQUATIONS OF FLUID MOTION
(Integral Form-Part III)

Combined Equation (First and Second Law of Thermodynamics)

Let us revisit the following equations derived in the previous section from the statements of first and second law of thermodynamics.

First law: \( \frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\left(dz\right) = -(Tds - \delta q) \)  \hspace{1cm} (2.4.1)

Second law: \( Tds_e - \delta q \geq 0; \) \( s_e \) :specific entropy

Combination of first and second law of thermodynamics leads to,

\[ -\left[ \frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\left(dz\right) \right] \geq 0 \] \hspace{1cm} (2.4.2)

Introducing the equality sign, Eq. (2.4.2) can be rewritten as,

\[ -\left[ \frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\left(dz\right) \right] = \delta (\text{loss}) = (Tds - \delta q) \] \hspace{1cm} (2.4.3)

The equality sign in Eq. (2.4.2) holds good for any steady and reversible (i.e. frictionless) while the inequality sign exists for all steady and irreversible flow involving friction. The extent to which the loss of useful/available energy occurs is mainly due to the irreversible flow phenomena including viscous effects. If some shaft work is involved, then the resulting equation becomes,

\[ -\left[ \frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\left(dz\right) \right] = \delta (\text{loss}) - \delta w_s \] \hspace{1cm} (2.4.4)

Both the Eqs (2.4.3 & 2.4.4) are valid for compressible and incompressible flows. In the case of frictionless and steady flows, the combined first and second law leads to the following equation;

\[ \frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\left(dz\right) = 0 \] \hspace{1cm} (2.4.5)
### Steady Flow Energy Equation (SFEE)

Let us recall the following energy equation derived in the previous section:

\[
\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt} \left[ \int_c \left( \dot{u} + \frac{1}{2} V^2 + gz \right) \rho dV \right] + \int_{cs} \left( \dot{h} + \frac{1}{2} V^2 + gz \right) \rho (\nabla \cdot \vec{n}) dA \tag{2.4.6}
\]

The general form of one-dimensional steady flow energy equation may be obtained from Eq. (2.4.6) and it has lot of engineering applications. If there is one inlet (section 1) and one outlet (section 2), then the first term in Eq. (2.4.6) can be omitted and the summation term in Eq. (2.4.6) reduces to single inlet and outlet.

\[
\dot{Q} - \dot{W}_s - \dot{W}_v = -\dot{m}_1 \left( \dot{h}_1 + \frac{1}{2} V_1^2 + gz_1 \right) + \dot{m}_2 \left( \dot{h}_2 + \frac{1}{2} V_2^2 + gz_2 \right) \tag{2.4.7}
\]

Since mass flow rate is constant, the continuity equation becomes \( \dot{m}_1 = \dot{m}_2 = \dot{m} \). So, the terms in Eq. (2.4.7) can be rearranged as follows;

\[
\left( \dot{h}_1 + \frac{1}{2} V_1^2 + gz_1 \right) = \left( \dot{h}_2 + \frac{1}{2} V_2^2 + gz_2 \right) - q + w_s + w_v
\]

or,

\[
\left( \dot{u} + \frac{p_1}{\rho} + \frac{1}{2} V_1^2 + gz_1 \right) = \left( \dot{u}_2 + \frac{p_2}{\rho} + \frac{1}{2} V_2^2 + gz_2 \right) - q + w_s + w_v \tag{2.4.8}
\]

Here, the terms \( q = \dot{Q}/m \); \( w_s = \dot{W}_s/m \); \( w_v = \dot{W}_v/m \) refer to heat and work transferred to the fluid per unit mass and \( H = \left( \dot{h} + \frac{1}{2} V^2 + gz \right) \) is the stagnation enthalpy. Eq. (2.4.8) is known as the **steady flow energy equation (SFEE)**. Each term in this equation has the dimensions of energy per unit mass. The other way to represent this equation is in the form **energy head** which is obtained by dividing both sides with the term \( g \) (i.e. acceleration due to gravity). So, the other form of Eq. (2.4.8) is given by,

\[
\frac{\dot{u}_1}{g} + \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{\dot{u}_2}{g} + \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 - h_q + h_s + h_v \tag{2.4.9}
\]

where, \( h_q = q/g \); \( h_s = w_s/g \); \( h_v = w_v/g \) are the head forms of heat and work transfers.

The terms \( \left( \frac{p}{\rho g} \right) \) and \( \left( \frac{V^2}{2g} \right) \) are called as **pressure head** and **velocity head**, respectively.
- A very common application of SFEE is the low-speed flows with no shaft work and negligible viscous dissipation such as liquid flow through pipes. In such cases, Eq. (2.4.9) may be written as,

\[
\frac{p_1 + \frac{V_1^2}{2g} + z_1}{\rho g} = \left( \frac{p_2 + \frac{V_2^2}{2g} + z_2}{\rho g} \right) + \left( \frac{\dot{u}_2 - \dot{u}_1 - q}{g} \right)
\]

or,

\[
h_{0,\text{in}} = h_{0,\text{out}} - h_f
\]

Here, the terms \(h_{0,\text{in}}\) and \(h_{0,\text{out}}\) are called as available/total head at the inlets and outlets, respectively and \(h_f\) is the loss in head due to friction.

**Bernoulli Equation**

Ignoring the frictional losses in steady flow energy equations, one can obtain the precise relation of pressure, velocity and elevation. This equation is called as *Bernoulli equation* developed in the year 1755. This equation is very famous and widely used with lot of restrictions. In general, all fluids are viscous and flows are associated with certain component of friction. In order to use *Bernoulli equation* correctly, one must confine the regions of flow which are nearly frictionless.

Consider an elemental fixed stream tube control volume of variable area \(A(s)\) and length \(ds\) as shown in Fig. 2.4.1. The fluid properties \(p, V\) and \(\rho\) vary along the streamline direction \(s\) and \(t\) while they are assumed to be uniform over the cross section \(A\). The streamtube is oriented at any arbitrary angle \(\theta\) with an elevation change \(dz = ds \sin \theta\).

![Fig. 2.4.1: Schematic representation of frictionless flow in a streamtube.](image)
Now, applying the principle of conservation of mass to this elemental control volume, one can write,

\[ \frac{d}{dt} \left( \int_{CV} \rho \, dV \right) + (\dot{m}_{out} - \dot{m}_{in}) = 0 \]

or, \( \frac{\partial \rho}{\partial t} \, dV + d\dot{m} = 0 \) \hspace{1cm} (2.4.11)

or, \( d\dot{m} = d \left( \rho AV \right) = -\frac{\partial \rho}{\partial t} \, Ads \)

The linear momentum equation can also be applied in the stream-wise direction i.e.

\[ \sum dF_s = \frac{d}{dt} \left( \int_{CV} \rho \, dV \right) + \left[ (\dot{m}V)_{out} - (\dot{m}V)_{in} \right] \]

or, \( \sum dF_s = \frac{\partial}{\partial t} (\rho V) \, Ads + d(\dot{m}V) \) \hspace{1cm} (2.4.12)

or, \( \sum dF_s = \frac{\partial}{\partial t} (VA)ds + \frac{\partial V}{\partial t} (\rho A)ds + \dot{m}dV + Vd\dot{m} \)

The elemental force \( (dF_s) \) consists of surface forces due to pressure \( (dF_{s,\text{pressure}}) \) and gravitational forces \( (dF_{s,\text{gravity}} = -dW \sin \theta = -\rho g 
A \sin \theta = -\rho g \, A \, dz) \) and its expression is given by,

\[ \sum dF_s = dF_{s,\text{pressure}} + dF_{s,\text{gravity}} = -Adp - \rho g \, Adz \] \hspace{1cm} (2.4.13)

Substitute Eq. (2.4.13) in the linear momentum equation (Eq. 2.4.12).

\[ -Adp - \rho g \, Adz = \frac{\partial}{\partial t} (VA)ds + \frac{\partial V}{\partial t} (\rho A)ds + \dot{m}dV + Vd\dot{m} \] \hspace{1cm} (2.4.14)
Recalling continuity equation, the first and last term of RHS of Eq. (2.4.14) cancels out. Divide both sides by $\rho A$ and rearrange it to obtain the final desired relation;

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V dV + g dz = 0$$  \hspace{1cm} (2.4.15)

This is the Bernoulli's equation for unsteady, frictionless flow along a streamline. It can be integrated between any two points ‘1’ and ‘2’ as given below;

$$\int_{1}^{2} \frac{\partial V}{\partial t} ds + \int_{1}^{2} \frac{dp}{\rho} + \frac{1}{2} (V_{2}^{2} - V_{1}^{2}) + g(z_{2} - z_{1}) = 0$$  \hspace{1cm} (2.4.16)

When the flow is unsteady ($\partial V/\partial t = 0$) and incompressible (constant-density), Eq. (2.1.16) reduces to,

$$\left(\frac{p_{2} - p_{1}}{\rho}\right) + \frac{1}{2} (V_{2}^{2} - V_{1}^{2}) + g(z_{2} - z_{1}) = 0$$

or, $\frac{p}{\rho} + \frac{1}{2} V_{1}^{2} + g z_{1} = \frac{p}{\rho} + \frac{1}{2} V_{2}^{2} + g z_{2}$

or, $\frac{p}{\rho} + \frac{1}{2} V_{2}^{2} + g z = \text{constant (along a streamline)}$

or, $\frac{p}{\rho g} + \frac{V_{2}^{2}}{2g} + z = h_{0}$

Eq. (2.4.17) is the Bernoulli equation for steady frictionless incompressible flow along a streamline. Many a times, the Bernoulli constant $(h_{0})$ is known as energy grade line and the height corresponding to pressure and elevation $(\frac{p}{\rho g} + z)$ is known as hydraulic grade line.
**Steady Flow Energy Equation vs Bernoulli Equation**

In general, the steady flow energy equation is applied to the control volumes with one-dimensional inlets and outlets. Often, in many situations, it is not strictly one-dimensional rather velocity may vary over the cross-section. So, the kinetic energy term in Eq. (2.4.6) can be modified by introducing a dimensionless correction factor \( \alpha \) so that the integral can be proportional to the square of the average velocity through the control surface for an incompressible flow.

\[
\int_{CS} \left( \frac{1}{2} V^2 \right) \rho (\vec{V} \cdot \hat{n}) dA = \alpha \left( \frac{1}{2} V_{avg}^2 \right) \hat{m}_i; \quad V_{avg} = \frac{1}{A} \int u dA \quad (2.4.18)
\]

If \( u \) is the velocity normal to the control surface, then the integral can be evaluated to obtain the expression of \( \alpha \) known as kinetic energy correction factor.

\[
\frac{1}{2} \rho \int u^3 dA = \frac{1}{2} \rho V_{avg}^3 A \\
\text{or, } \alpha = \frac{1}{A} \int \left( \frac{u}{V_{avg}} \right)^3 dA \quad (2.4.19)
\]

So, the general form of steady flow energy equation for an incompressible flow can be obtained from Eq. (2.4.8) by using the parameter \( \alpha \).

\[
\frac{p_1}{\rho} + \frac{\alpha_1 V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{\alpha_2 V_2^2}{2} + gz_2 + \left( \hat{u}_2 - \hat{u}_1 - q \right) + w_s + w_v \quad (2.4.20)
\]

This relation (Eq. 2.2.20) involves the terms that accounts for friction, heat transfer, shaft work and any viscous dissipation. In contrast, the strict restrictions are imposed Bernoulli’s equation (Eq. 2.4.17) that can be listed as follows;

- Steady, incompressible and frictionless flow
- Flow along a single streamline because different streamlines may have different Bernoulli constant.
- Flow with one inlet and outlet
- No shaft work and heat transfer between the sections.
Module 2 : Lecture 5
GOVERNING EQUATIONS OF FLUID MOTION
(Differential Form-Part I)

In general, there are two broad paths by which the fluid motion can be analyzed. The first case uses the estimates of gross effects of parameters involved in the basic laws over a finite region/control volume. They have been discussed in the previous sections. In the other one, the flow patterns are analyzed point-by-point in an infinitesimal region and the basic differential equations are developed by satisfying the basic conservation laws.

Concept of Material Derivative

The time and space derivative applied to any fluid property can be represented in mathematical form and called as substantial/ material/total time derivative. The Lagrangian frame follows the moving position of individual particles while the coordinate systems are fixed in space, in case of Eulerian frame of reference and hence, it is commonly used. Let us illustrate the concept of material derivative through velocity field. In Eulerian system, the Cartesian form of velocity vector field is defined as,

\[ \vec{V}(r,t) = \hat{i}u(x, y, z, t) + \hat{j}v(x, y, z, t) + \hat{k}w(x, y, z, t) \]  \hspace{1cm} (2.5.1)

Using Newton’s second law motion, for an infinitesimal fluid system, the acceleration vector field \( a \) for the flow can be computed

\[ \vec{a} = \frac{d\vec{V}}{dt} = \hat{i} \frac{du}{dt} + \hat{j} \frac{dv}{dt} + \hat{k} \frac{dw}{dt} \]  \hspace{1cm} (2.5.2)

Each scalar component of \( u, v \) and \( w \) is a function of four variables \( (x, y, z, t) \) and also, \( u = \frac{dx}{dt} \); \( v = \frac{dy}{dt} \) and \( w = \frac{dz}{dt} \). So, the scalar time derivative is obtained as,

\[ \frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \]  \hspace{1cm} (2.5.3)
The compact form of total derivative of $u, v$ and $w$ are written as,

$$
\frac{du}{dt} = \frac{\partial u}{\partial t} + \left( \vec{V} \cdot \nabla \right) u; \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + \left( \vec{V} \cdot \nabla \right) v; \quad \frac{dw}{dt} = \frac{\partial w}{\partial t} + \left( \vec{V} \cdot \nabla \right) w
$$

(2.5.4)

The compact dot product involving $\left( \vec{V} \right)$ and gradient operator $\left( \nabla \right)$ is defined as,

$$
\left( \vec{V} \cdot \nabla \right) = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}; \quad \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}
$$

(2.5.5)

The total acceleration is obtained as,

$$
\ddot{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left( u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} \right) = \frac{\partial \vec{V}}{\partial t} + \left( \vec{V} \cdot \nabla \right) \vec{V}
$$

(2.5.6)

The term $\left( \frac{\partial \vec{V}}{\partial t} \right)$ is called the \textit{local acceleration} and it vanishes when the flow is steady. The other one in the bracket is called the \textit{convective acceleration} which arises when there is a spatial velocity gradient. The combination of these two is called as \textit{substantial/ material/total time} derivative. This concept can be extended to any scalar/vector flow variable. Similar expression can be written for pressure and temperature as well.

$$
\frac{dp}{dt} = \frac{\partial p}{\partial t} + \left( u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} \right) = \frac{\partial p}{\partial t} + \left( \vec{V} \cdot \nabla \right) p
$$

$$
\frac{dT}{dt} = \frac{\partial T}{\partial t} + \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = \frac{\partial T}{\partial t} + \left( \vec{V} \cdot \nabla \right) T
$$

(2.5.7)
Mass Conservation Equation

Let us consider an infinitely small elemental control volume having dimensions $dx$, $dy$ and $dz$ as shown in Fig. 2.5.1. The flow through each side of the element may be treated as one-dimensional and continuum concept may be retained. It leads to the fact that all the fluid properties can be considered to be varying uniformly as a function of time and position.

![Elemental control volume with inlet and outlet mass flow.](image)

The basic control volume relations discussed earlier can be applied here and it takes in the following form.

$$
0 = \iiint_{c v} \left( \sum_{i} \rho A_i V_i \right)_{out} - \sum_{i} \left( \rho A_i V_i \right)_{in} - \int_{c v} \frac{\partial \rho}{\partial t} \frac{dV}{dt} - \sum_{i} \left[ \frac{\partial}{\partial x} \left( \rho u \right) dA \right] dy \, dz
$$

(2.5.8)

The element being very small, the volume integral is reduced to the following differential form,

$$
\int_{c v} \frac{\partial \rho}{\partial t} \frac{dV}{dt} = \frac{\partial \rho}{\partial t} \iiint dx \, dy \, dz
$$

(2.5.9)
The mass flow terms appear in all six faces with three inlets and three outlets. As shown in Fig. 2.5.1, these terms can be summarized in the following table.

<table>
<thead>
<tr>
<th>Face</th>
<th>Inlet mass flow</th>
<th>Outlet mass flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>( \rho u , dy , dz )</td>
<td>( \left[ \rho u + \frac{\partial}{\partial x} (\rho u) \right] dx , dy , dz )</td>
</tr>
<tr>
<td>y</td>
<td>( \rho v , dx , dz )</td>
<td>( \left[ \rho v + \frac{\partial}{\partial y} (\rho v) \right] dx , dz )</td>
</tr>
<tr>
<td>z</td>
<td>( \rho w , dx , dy )</td>
<td>( \left[ \rho w + \frac{\partial}{\partial z} (\rho w) \right] dx , dy )</td>
</tr>
</tbody>
</table>

After substituting these terms in Eq. (2.5.8), one can get,

\[
\frac{\partial \rho}{\partial t} dx \, dy \, dz + \frac{\partial}{\partial x} (\rho u) dx \, dy \, dz + \frac{\partial}{\partial y} (\rho v) dx \, dy \, dz + \frac{\partial}{\partial z} (\rho w) dx \, dy \, dz = 0
\]

or,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \tag{2.5.10}
\]

or,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0
\]

This is the desired form of mass conservation equation for an infinitesimal control volume in Cartesian coordinate system. It is applicable for major categories of flows such as steady/unsteady, inviscid/viscous, incompressible/compressible. Many a times, it is referred as equation of continuity because it requires no assumptions except the fact that density and velocity are continuous functions. Alternatively, the continuity equation is also expressed in cylindrical coordinate system which is useful in many practical flow problems. In this case, any arbitrary point is defined by the coordinates \((r \, \theta \, z)\) where, \(z\) is the distance along \(z\)-axis, \(r\) is the radial distance and \(\theta\) is the rotational angle about the axis as shown in Fig. 2.5.2. Then, the conversion is possible using the transformation as given below,

\[
r = \left( x^2 + y^2 \right)^{1/2} ; \quad \theta = \tan^{-1} \left( \frac{y}{x} \right) ; \quad z = z
\]
Thus, the general continuity equation in cylindrical coordinate system becomes,

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \]  

(2.5.12)

- If the flow is steady, then all the properties are functions of position only. So the Eqs (2.5.10 & 2.5.12) reduces to,

\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \]  

(2.5.13)

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \]

- In a special case, if the flow is incompressible, then density changes are negligible i.e. \( \frac{\partial \rho}{\partial t} = 0 \), regardless of whether the flow is steady or not. So, the Eq. (2.5.13) is still valid without the density term.

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \]  

(2.5.14)

Vector relation : \( \nabla \cdot \vec{V} = 0 \)
Stress Field

With respect to Newton’s laws of motion, force must be applied to induce acceleration on a body. Since, the fluid is treated as ‘continuum’, one must understand the types of forces that act on the fluid particles. In general, each fluid particle experiences surface forces (i.e. pressure, friction) and body forces (i.e. gravitation). The surface forces are generated by their contacts with other fluid particles and solid medium, leading to stresses. The body forces are experienced throughout the particle and the gravitational body force per unit volume is quantified as $\rho g$, where $\rho$ is the density and $g$ is the gravitational acceleration.

The concept of stress describes the way in which the surface forces acting on the fluid and solid boundaries are transmitted into the medium. In a solid, the stresses are induced within the body. In the case of fluids, when a body moves through a fluid, stresses are developed within the fluid. Consider the contact force generated between fluid particles when the surface of a fluid particle in contact with other (Fig. 2.5.3-a). If a portion of the surface $\delta A$ is considered at some point ‘$P$’, the orientation of $\delta A$ is given by the unit vector $\hat{n}$ drawn normal to the particle outward. The force $\delta F$ acting on $\delta A$ can be resolved into two components; normal to the area and tangent to the area (Fig. 2.5.3-b). The stresses are then quantified with respect to this force per unit area. Thus, the normal stress ($\sigma_n$) and shear stress ($\tau_n$) are then defined as below;

$$\sigma_n = \lim_{\delta A_n \to 0} \left( \frac{\delta F_n}{\delta A_n} \right); \quad \tau_n = \lim_{\delta A_n \to 0} \left( \frac{\delta F_t}{\delta A_n} \right) \quad (2.5.15)$$

Since fluid is treated as ‘continuum’, it is possible to resolve these forces around the point ‘$P$’ to get different stresses around that point. In rectangular coordinates, the stressed can be considered to act on the planes drawn as outward normal in the respective $x$, $y$ and $z$ directions (Fig. 2.5.3-c). Then, Eq. (2.5.15) can written for $x$-direction as,

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \left( \frac{\delta F_x}{\delta A_x} \right); \quad \tau_{yx} = \lim_{\delta A_y \to 0} \left( \frac{\delta F_y}{\delta A_x} \right); \quad \tau_{zx} = \lim_{\delta A_z \to 0} \left( \frac{\delta F_z}{\delta A_x} \right) \quad (2.5.16)$$
Here, the first subscript indicates the plane on which the stress act (i.e., plane is perpendicular to $x$-axis) and the second subscript denotes the direction of the stress. Although, there may be infinite number of planes passing through the point ‘$P$’, but we shall consider only on orthogonal planes mutually perpendicular to each other. Hence, the stress at any point is specified by nine components in the form of matrix as given below, where $\sigma$ denotes the normal stress and $\tau$ is referred as shear stress.

$$
\text{Stress at a point, } \\
\begin{pmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{pmatrix}
$$

Referring to an infinitesimal element of size $\Delta x, \Delta y$ and $\Delta z$ as shown in Fig. 2.5.3-d, there are six planes on which stresses act. The planes are named and denoted as positive or negative according to the direction of outward normal to the plane. A stress component is ‘positive, when the direction of stress component and plane on which it acts are both positive or both negative. All the stresses shown in Fig. 2.5.3-d, are all positive.

Fig. 2.5.3: Description of stress in a flow field: (a) Concept of stress; (b) Normal and shear stress; (c) Force and stress components in orthogonal coordinates; (d) Notation of stress.
Module 2 : Lecture 6
GOVERNING EQUATIONS OF FLUID MOTION
(Differential Form-Part II)

Linear Momentum Equation (Differential Form)

Recall the one-dimensional control-volume equation for linear momentum;

$$\sum F = \frac{d}{dt}(\int_{cv} \rho \mathbf{v} dV) + \left[ \sum \left( m_i \mathbf{v}_i \right)_{out} - \sum \left( m_i \mathbf{v}_i \right)_{in} \right]$$  \hspace{1cm} (2.6.1)

When this equation is applied to the elemental control volume shown in Fig. 2.6.1, the volume integral derivative (second term of RHS of Eq. 2.6.1) reduces to,

$$\frac{\partial}{\partial t} \left( \int_{cv} \rho \mathbf{V} dV \right) \approx \frac{\partial}{\partial t} (\rho \mathbf{V}) dx dy dz$$  \hspace{1cm} (2.6.2)

Fig. 2.6.1: Elemental control volume with inlet and outlet momentum flux.
The momentum flux terms appear in all six faces with three inlets and three outlets. As shown in Fig. 2.6.1, these terms can be summarized in the following table.

<table>
<thead>
<tr>
<th>Face</th>
<th>Inlet mass flow</th>
<th>Outlet mass flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$(\rho u \vec{V}) dy , dz$</td>
<td>$\left[ \rho u \vec{V} + \frac{\partial}{\partial x} (\rho u \vec{V}) \right] dx , dy , dz$</td>
</tr>
<tr>
<td>$y$</td>
<td>$(\rho v \vec{V}) dx , dz$</td>
<td>$\left[ \rho v \vec{V} + \frac{\partial}{\partial y} (\rho v \vec{V}) \right] dy , dz$</td>
</tr>
<tr>
<td>$z$</td>
<td>$(\rho w \vec{V}) dx , dy$</td>
<td>$\left[ \rho w \vec{V} + \frac{\partial}{\partial z} (\rho w \vec{V}) \right] dx , dz$</td>
</tr>
</tbody>
</table>

After substituting these terms in RHS of Eq. (2.6.1), and using Eq. (2.6.2), one can get,

$$
\sum \vec{F} = \left[ \frac{\partial}{\partial t} (\rho \vec{V}) + \frac{\partial}{\partial x} (\rho u \vec{V}) + \frac{\partial}{\partial y} (\rho v \vec{V}) + \frac{\partial}{\partial z} (\rho w \vec{V}) \right] dx \, dy \, dz \quad (2.6.3)
$$

Now, let us split and simplify the terms in RHS of Eq. (2.6.3);

$$
\frac{\partial}{\partial t} (\rho \vec{V}) + \frac{\partial}{\partial x} (\rho u \vec{V}) + \frac{\partial}{\partial y} (\rho v \vec{V}) + \frac{\partial}{\partial z} (\rho w \vec{V}) = \vec{V} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] + \rho \left[ \frac{\partial \vec{V}}{\partial t} + u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} \right]
$$

or,

$$
\frac{\partial}{\partial t} (\rho \vec{V}) + \frac{\partial}{\partial x} (\rho u \vec{V}) + \frac{\partial}{\partial y} (\rho v \vec{V}) + \frac{\partial}{\partial z} (\rho w \vec{V}) = \vec{V} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] + \rho \frac{d\vec{V}}{dt}
$$

(2.6.4)

The first part in the RHS of Eq. (2.6.4) is the continuity equation and vanishes while the second part is the total acceleration of the fluid particle. So, the Eq. (2.6.3) reduces to

$$
\sum \vec{F} = \rho \frac{d\vec{V}}{dt} dx \, dy \, dz
$$

or,

$$
\frac{d\vec{F}_{\text{gravity}}}{dt} + \frac{d\vec{F}_{\text{surface}}}{dt} = \rho \frac{d\vec{V}}{dt} dx \, dy \, dz
$$

(2.6.5)

or,

$$
\left( \frac{d\vec{F}}{dt} \right)_{\text{gravity}} + \left( \frac{d\vec{F}}{dt} \right)_{\text{surface}} = \rho \frac{d\vec{V}}{dt} \quad (\text{since } d\vec{V} = dx \, dy \, dz)
$$
Since the fluid element is very small, the summation of forces can be represented in differential form as given in Eq. (2.6.5). Here, net force on the control volume is of two types; body forces and surface forces. The first one is mainly due to gravity that acts on entire fluid element. This gravity force per unit volume may be represented as,

$$\left( \frac{d\vec{F}}{dV} \right)_{\text{gravity}} = \rho \vec{g}$$  \hspace{1cm} (2.6.6)

The surface forces mainly acts on the sides of control surface and is the sum of contribution from hydrostatic pressure and viscous stresses. The hydrostatic pressure acts normal to the surface while the viscous stresses ($\tau_{ij}$) arise due to the velocity gradient. Referring to the notations given in Fig. 2.6.2-a, the sum of these stresses can be represented as a stress tensor ($\sigma_{ij}$) as follows;

$$\sigma_{ij} = \begin{pmatrix}
-p + \tau_{xx} & \tau_{yx} & \tau_{xz} \\
\tau_{xy} & -p + \tau_{yy} & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & -p + \tau_{zz}
\end{pmatrix}$$  \hspace{1cm} (2.6.7)

Fig. 2.6.2: Control volume showing the notation of stresses and surface forces.
It may be noted that the gradient in stresses produces the net force on the control surface not the stresses. So, Fig. 2.6.2-b, the net surface force per unit volume in $x$-direction can be calculated as,

$$dF_{x,\text{surface}} = \left[ \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx}) \right] dx\,dy\,dz$$

or,

$$\frac{dF_x}{dV} = \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{sx}) + \frac{\partial}{\partial y}(\tau_{sy}) + \frac{\partial}{\partial z}(\tau_{sz})$$  \hspace{1cm} (2.6.8)$$

In a similar manner, the net surface forces per unit volume in $y$ and $z$ directions are calculated as,

$$\frac{dF_y}{dV} = \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{sy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{sz})$$

and,

$$\frac{dF_z}{dV} = \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{sz}) + \frac{\partial}{\partial y}(\tau_{sz}) + \frac{\partial}{\partial z}(\tau_{zz})$$  \hspace{1cm} (2.6.9)$$

In the vector form, the Eqs (2.6.8) and (2.6.9) are represented as,

$$\frac{d\vec{F}}{dV}_{\text{surface}} = -\nabla p + \frac{d\vec{F}}{dV}_{\text{viscous}} = -\nabla p + \nabla \cdot \mathbf{\tau}_{ij}$$  \hspace{1cm} (2.6.10)$$

where,

$$\frac{d\vec{F}}{dV}_{\text{viscous}} = \hat{i} \left( \frac{\partial \tau_{sx}}{\partial x} + \frac{\partial \tau_{sy}}{\partial y} + \frac{\partial \tau_{sz}}{\partial z} \right) + \hat{j} \left( \frac{\partial \tau_{sx}}{\partial x} + \frac{\partial \tau_{sy}}{\partial y} + \frac{\partial \tau_{sz}}{\partial z} \right) + \hat{k} \left( \frac{\partial \tau_{sx}}{\partial x} + \frac{\partial \tau_{sy}}{\partial y} + \frac{\partial \tau_{sz}}{\partial z} \right)$$

or,

$$\frac{d\vec{F}}{dV}_{\text{viscous}} = \nabla \cdot \mathbf{\tau}_{ij}$$

and,

$$\mathbf{\tau}_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{yx} & \tau_{yy} & \tau_{zy} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$  \hspace{1cm} (2.6.11)$$

Now, the Eqs (2.6.5, 2.6.6 & 2.6.10) can be combined to obtain the differential equation for linear momentum.

$$\rho \ddot{\mathbf{V}} - \nabla p + \nabla \cdot \mathbf{\tau}_{ij} = \rho \frac{d\vec{V}}{dt}$$  \hspace{1cm} (2.6.12)$$
Navier-Stokes Equation

The differential equation for linear momentum is valid for any general motion where the any particular fluid is characterized by its corresponding viscous-stress terms. The vector form of Eq. (2.6.12) can be written in the scalar form as follow:

\[
\begin{align*}
\text{x-Momentum : } & \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\text{y-Momentum : } & \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\
\text{z-Momentum : } & \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{align*}
\]

(2.6.13)

It may be noted that the last three convective terms on the RHS of Eq. (2.6.13) makes it highly non-linear and complicates the general analysis. A simplification is possible for considering an incompressible flow of Newtonian fluid where the viscous stresses are proportional to the element strain rate and coefficient of viscosity (\(\mu\)). For an incompressible flow, the shear terms may be written as,

\[
\begin{align*}
\tau_{xx} &= 2\mu \frac{\partial u}{\partial x}; \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y}; \quad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}; \quad \tau_{xy} = \tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial \tau_{xx}}{\partial x} \right); \quad \tau_{xz} = \tau_{zx} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial \tau_{xx}}{\partial x} \right) \quad (2.6.14)
\end{align*}
\]

Thus, the differential momentum equation for Newtonian fluid with constant density and viscosity is given by,

\[
\begin{align*}
\text{x-Momentum : } & \rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \frac{du}{dt} \\
\text{y-Momentum : } & \rho g_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = \rho \frac{dv}{dt} \\
\text{z-Momentum : } & \rho g_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \rho \frac{dw}{dt}
\end{align*}
\]

(2.6.15)

It is a second-order, non-linear partial differential equation and is known as Navier-Stokes equation. In vector form, it may be represented as,

\[
\rho \ddot{\mathbf{V}} - \nabla p + \mu \nabla^2 \mathbf{V} = \rho \frac{d\mathbf{V}}{dt}
\]

(2.6.16)
This equation has four unknowns \( p, u, v, \) and \( w \) and must be combined with continuity relation to obtain complete information of the flow field.

**Euler’s Equation**

When the viscous stresses components in the general form of linear momentum differential equation are neglected \( (\tau_g = 0) \), then vector Eq. (2.6.12) reduce to the following form;

\[
\rho \ddot{V} - \nabla p = \rho \frac{d\dot{V}}{dt}
\]

(2.6.16)

The same equation in scalar form is written as,

\[
\begin{align*}
\text{x-Momentum:} & \quad \rho g_x - \frac{\partial p}{\partial x} = \rho \frac{du}{dt} \\
\text{y-Momentum:} & \quad \rho g_y - \frac{\partial p}{\partial y} = \rho \frac{dv}{dt} \\
\text{z-Momentum:} & \quad \rho g_z - \frac{\partial p}{\partial z} = \rho \frac{dw}{dt}
\end{align*}
\]

(2.6.17)

This relation is valid for frictionless flow and known as the *Euler’s equation for inviscid flow*.

**Bernoulli’s Equation**

In the previous section, the *Bernoulli’s equation* was derived from the steady flow energy equation by ignoring the frictional losses. In the same line, the linear momentum equation reduces *Euler’s equation* when the viscous stress components are neglected which is true only when the flow is *irrotational and frictionless*. A flow is said to be *irrotational* when there is no vorticity \( (\vec{\xi} = 0) \) or angular velocity \( (\vec{\omega} = 0) \). Mathematically, it represented as below;

\[
\vec{\xi} = 2\vec{\omega} = \text{curl } \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0
\]

(2.6.18)
Now, rewrite Euler’s equation in the following form,

$$\rho \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = \rho \ddot{g} - \nabla p$$  \hspace{1cm} (2.6.19)

There exists a vector identity to simplify the second term of LHS of Eq. (2.6.19);

$$\left( \vec{V} \cdot \nabla \right) \vec{V} = \nabla \left( \frac{1}{2} \vec{V}^2 \right) + \left( \vec{\xi} \times \vec{V} \right)$$  \hspace{1cm} (2.6.20)

So, Eq. (2.6.19) can be again rewritten as,

$$\frac{\partial \vec{V}}{\partial t} + \nabla \left( \frac{1}{2} \vec{V}^2 \right) + \left( \vec{\xi} \times \vec{V} \right) + \frac{\nabla p}{\rho} - \ddot{g} = 0$$  \hspace{1cm} (2.6.21)

Take the dot product of the entire Eq. (2.6.21) with an arbitrary vector displacement $d\vec{r}$.

$$\left[ \frac{\partial \vec{V}}{\partial t} + \nabla \left( \frac{1}{2} \vec{V}^2 \right) + \left( \vec{\xi} \times \vec{V} \right) + \frac{\nabla p}{\rho} - \ddot{g} \right] \cdot d\vec{r} = 0$$  \hspace{1cm} (2.6.22)

Let us assume that $\left( \vec{\xi} \times \vec{V} \right) \cdot d\vec{r} = 0$, which is true under the following conditions;

- When there is no flow i.e. $\vec{V} = 0$ (hydrostatic case)
- When the flow is irrotational, i.e. $\vec{\xi} = 0$.
- $d\vec{r}$ is perpendicular to $\left( \vec{\xi} \times \vec{V} \right)$ which is a very rare case.
- $d\vec{r}$ is parallel to $\vec{V}$ so that one can go along the streamline.

Now, use the condition given by Eq. (2.6.18), when the flow is irrotational and take $\ddot{g} = -g\hat{k}$ so that Eq. (2.6.22) reduces to,

$$\frac{\partial \vec{V}}{\partial t} \cdot d\vec{r} + d \left( \frac{1}{2} \vec{V}^2 \right) + \frac{dp}{\rho} + g\hat{k} = 0$$  \hspace{1cm} (2.6.23)
Integrate Eq. (2.6.23) along a streamline between two points ‘1 and 2’ for a frictionless compressible flow.

\[
\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2} (V_z^2 - V_i^2) + g(z_2 - z_i) = 0 \tag{2.6.24}
\]

where, \( ds \) is the arc length along the streamline. Eq. (2.6.24) is known as the Bernoulli’s equation for frictionless unsteady flow along a streamline. Again if the flow is incompressible (\( \rho = \) constant), and steady \( \left( \frac{\partial}{\partial t} = 0 \right) \), the Eq. (2.6.24) reduces to,

\[
\frac{p_i}{\rho} + \frac{1}{2} V_i^2 + g z_i = \frac{p_2}{\rho} + \frac{1}{2} V_2^2 + g z_2 \tag{2.6.25}
\]

or, \[ \frac{p}{\rho} + \frac{1}{2} V^2 + g z = \text{constant along a streamline} \]

This equation is same as the one derived from steady flow energy equation and true only for frictionless, incompressible, irrotational and steady flow along a streamline.
Module 2 : Lecture 7
GOVERNING EQUATIONS OF FLUID MOTION
(Differential Form-Part III)

Energy Equation (Differential Form)

Recall the integral relation of energy equation for a fixed control volume;

\[
\dot{Q} - \dot{W} - \dot{\dot{W}}_v = \frac{\partial}{\partial t} \left( \int_{CV} e \rho dV \right) + \int_{CS} \left( e + \frac{p}{\rho} \right) \rho (\vec{V} \cdot \vec{n}) dA
\]  

(2.7.1)

Fig. 2.7.1: Elemental control volume showing heat flow and viscous work rate in x-direction.

If the control volume happens to be an elemental system as shown in Fig. 2.7.1(a), then there will be no shaft work term \( \dot{W}_s = 0 \). Denoting the energy per unit volume as \( e = \dot{u} + \frac{1}{2} \dot{V}^2 + gz \), the net energy flow across the six control surface can be calculated from the following table;

<table>
<thead>
<tr>
<th>Face</th>
<th>Inlet energy flow</th>
<th>Outlet energy flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \rho u \left( e + \frac{p}{\rho} \right) ) ( dy ) ( dz )</td>
<td>( \rho u \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial x} \left( \rho u \right) \left( e + \frac{p}{\rho} \right) dx ) ( dy ) ( dz )</td>
</tr>
<tr>
<td>( y )</td>
<td>( \rho v \left( e + \frac{p}{\rho} \right) ) ( dx ) ( dz )</td>
<td>( \rho v \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial y} \left( \rho v \right) \left( e + \frac{p}{\rho} \right) dy ) ( dx ) ( dz )</td>
</tr>
<tr>
<td>( z )</td>
<td>( \rho w \left( e + \frac{p}{\rho} \right) ) ( dx ) ( dy )</td>
<td>( \rho w \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial z} \left( \rho w \right) \left( e + \frac{p}{\rho} \right) dz ) ( dx ) ( dy )</td>
</tr>
</tbody>
</table>
Hence, Eq. (2.7.1) can be written in the following form;

$$\dot{Q} - \dot{W}_c = \left[ \frac{\partial}{\partial t} \rho \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial x} (\rho u) \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial y} (\rho v) \left( e + \frac{p}{\rho} \right) + \frac{\partial}{\partial z} (\rho w) \left( e + \frac{p}{\rho} \right) \right] dx dy dz$$

(2.7.2)

With the help of continuity equation and similar analogy considered during the derivation of momentum equation, Eq. (2.7.2) takes the following form;

$$\dot{Q} - \dot{W}_c = \left[ \rho \frac{de}{dt} + \vec{v} \cdot (\nabla p) \right] dx dy dz$$

(2.7.3)

If one considers the energy transfer as heat \(\dot{Q}\) through pure conduction, the Fourier’s law of heat conduction can be applied to the elemental system.

$$\vec{q} = -k \nabla T$$

(2.7.4)

where, \(k\) is the thermal conductivity of the fluid. The heat flow passing through \(x\)-face is shown in Fig. 2.7.1(b) and for all the six faces, it is summarized in the following table;

<table>
<thead>
<tr>
<th>Face</th>
<th>Inlet heat flux</th>
<th>Outlet heat flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(q_x dy dz)</td>
<td>(q_x + \frac{\partial}{\partial x} (q_x) dx dy dz)</td>
</tr>
<tr>
<td>(y)</td>
<td>(q_y dx dz)</td>
<td>(q_y + \frac{\partial}{\partial y} (q_y) dy dx dz)</td>
</tr>
<tr>
<td>(z)</td>
<td>(q_z dx dy)</td>
<td>(q_z + \frac{\partial}{\partial z} (q_z) dz dx dy)</td>
</tr>
</tbody>
</table>

The net heat flux can be obtained by the difference in inlet and outlet heat fluxes;

$$\dot{Q} = - \left[ \frac{\partial}{\partial x} (q_x) + \frac{\partial}{\partial y} (q_y) + \frac{\partial}{\partial z} (q_z) \right] dx dy dz = - (\nabla \cdot \vec{q}) dx dy dz = \nabla \cdot (k \nabla T) dx dy dz$$

(2.7.5)

The rate of work done by the viscous stresses on the left \(x\)-face as shown in Fig. 2.7.1(b) is given by,

$$\dot{W}_{v,LF} = -w_x dy dz = -(u_{xx} + v_{xy} + w_{xz}) dy dz$$

(2.7.6)
In the similar manner, the net viscous rates are obtained and is given by,

$$\dot{W}_v = -\nabla \cdot (\tau \cdot \nabla) \, dx \, dy \, dz$$

(2.7.7)

Now, substitute Eqs. (2.7.5 & 2.7.7) in Eq. (2.7.3),

$$\rho \frac{de}{dt} + \dot{V} \cdot \nabla p = \nabla \cdot (k \nabla T) + \nabla \cdot (\dot{V} \cdot \tau)$$

(2.7.8)

The second term in the RHS of Eq. (2.7.8) can be written in the following form;

$$\nabla \cdot (\dot{V} \cdot \tau) = \dot{V} \cdot (\nabla \cdot \tau) + \Phi$$

(2.7.9)

Here, $\Phi$ is known as the viscous-dissipation function. For, Newtonian incompressible viscous fluid, this function as the following form;

$$\Phi = \mu \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right]$$

(2.7.10)

Since all the terms in Eq. (2.7.10) are quadratic, so the viscous dissipation terms are always positive i.e. the flow always tends to lose its available energy due to dissipation.

- When Eq. (2.7.9) is used in Eq. (2.7.8), simplified using linear-momentum equation and the terms are rearranged, then the general form of energy equation is obtained for Newtonian-viscous fluid.

$$\rho \frac{d\dot{u}}{dt} + p \nabla \cdot \dot{V} = \nabla \cdot (k \nabla T) + \Phi$$

(2.7.11)

- For analysis point of view, the following valid approximations can be made for Eq. (2.7.11) i.e. $d\dot{u} \approx c_v dT$; $c_v, \mu, k$ and $\rho$ are constants.

$$\rho c_v \frac{dT}{dt} = k \nabla^2 T + \Phi$$

where $\frac{dT}{dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$

(2.7.12)
Boundary/Initial Conditions for Basic Equations

The three basic differential equations of fluid motion may be summarized as follows;

Continuity: \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \)

Momentum: \( \rho \ddot{\vec{V}} - \nabla p + \nabla \cdot \tau_{ij} = \rho \frac{d\vec{V}}{dt} \) \hspace{2cm} (2.7.13)

Energy: \( \rho \frac{d\dot{u}}{dt} + p \left( \nabla \cdot \vec{V} \right) = \nabla \cdot (k \nabla T) + \Phi \)

In general, there are five unknowns namely, \( \rho, \vec{V}, p, \dot{u} \) and \( T \) in these three equations. Additional two relations can be obtained from any thermodynamic state of the fluid. For example,

\[ p = p(\rho, T) \quad \text{and} \quad \dot{u} = \dot{u}(\rho, T) \] \hspace{2cm} (2.7.14)

For perfect gas with constant specific heats, Eq. (2.7.14) may be written as,

\[ \rho = \frac{p}{RT} \quad \text{and} \quad \dot{u} = \int c_v dT \] \hspace{2cm} (2.7.15)

The solution of above differential equation needs initial conditions if the flow is unsteady i.e. spatial distribution of each variable at different time steps. In other words, at \( t = 0 \), the flow properties \( \rho, \vec{V}, p, \dot{u} \) and \( T \) are known functions of \( f(x, y, z) \) and these variables need to be analyzed at all time steps.

The three boundaries that a fluid commonly encounters during the flow are classified as solid wall, inlet/outlet and interface as shown in Fig. 2.7.2. Let us discuss them one by one;

![Fig. 2.7.2: Typical boundary conditions for viscous fluid flow analysis.](image)
- For a solid impermeable wall, there is no-slip and no-temperature jump condition which is stated as \( \vec{V}_{\text{fluid}} = \vec{V}_{\text{wall}} \) and \( T_{\text{fluid}} = T_{\text{wall}} \). In the case of adiabatic wall, \( \left( k \frac{\partial T}{\partial n} \right)_{\text{fluid}} = q_{\text{solid}} \).

- At inlet and outlet section of the flow, the complete distribution of \( \vec{V}, p \) and \( T \) must be known at all the times.

- At the liquid-gas interface, one can realize the situations such as equality in the vertical velocity \( w_{\text{liquid}} = w_{\text{gas}} \), viscous-shear stress \( \tau_{zy,\text{liquid}} = \tau_{zy,\text{gas}} \), \( \tau_{zx,\text{liquid}} = \tau_{zx,\text{gas}} \) and heat transfer \( q_{z,\text{liquid}} = q_{z,\text{gas}} \). If the upper fluid (gas) happens to be an atmosphere i.e. \( p_{\text{liquid}} \approx p_{\text{atm}} \), then it is called as free surface flow (e.g. open-channel flow).

**Simplification of Basic Equations**

The governing equations of the fluid motion (Eq. 2.7.13) must be solved simultaneously in certain domain with appropriate boundary and initial conditions to obtain the solution of complete flow field. In many practical cases, simplification of basic equation is possible with certain assumption that reduces the mathematical complications. Here we shall discuss two such cases;

**Case I:** If the flow is incompressible and the properties such as density \( \rho \), viscosity \( \mu \) and thermal conductivity \( k \) are assumed to be constant, then Eq. (2.7.13) reduces to the following form;

\[
\begin{align*}
\text{Continuity:} & \quad \nabla \cdot \vec{V} = 0 \\
\text{Momentum:} & \quad \rho \ddot{\vec{V}} - \nabla p + \mu \nabla^2 \vec{V} = \rho \frac{d\vec{V}}{dt} \\
\text{Energy:} & \quad \rho c_v \frac{dT}{dt} = k \nabla^2 T + \Phi
\end{align*}
\] (2.7.16)

Since \( \rho \) is constant, it leads to three unknowns \( p, \vec{V} \) and \( T \) in Eq. (2.7.16). Moreover, the *continuity and momentum* equations are independent of \( T \). So, they can be solved simultaneously for \( p \) and \( \vec{V} \), respectively. The typical boundary conditions are, known values of \( p \) and \( \vec{V} \) at inlet and outlet, \( \vec{V} = \vec{V}_{\text{wall}} \) (solid surface) and \( p \approx p_{\text{atm}} \) (free surface). The energy equation must be solved separately \( T \) and this particular case is known as *thermal decoupling*. 
Case II: When the flow is inviscid throughout for which $\mu = 0$, the momentum equation reduces to Euler’s equation. Further, it can be integrated along a streamline to obtain the Bernoulli’s equation. In such cases, no slip boundary conditions at the wall are always maintained. It allows the flow to be always parallel to the wall but does not allow the flow into the wall i.e. the normal velocities must match $(\hat{V}_n)_{\text{fluid}} = (\hat{V}_n)_{\text{wall}}$. Most of the cases, the wall is fixed, so that $(\hat{V}_n) = 0$. 